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ERGODIC GROUP ACTIONS WITH NONUNIQUE INVARIANT MEANS

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ABSTRACT. Let $M(X,G)$ be the set of G-invariant means on $L^\infty(X,\mathcal{B},P)$, where G is a countable group acting ergodically as measure preserving transformations on a nonatomic probability space $(X,\mathcal{B},P)$. We show that if there exists $\mu \in M(X,G)$, $\mu \neq P$, then $M(X,G)$ contains an isometric copy of $\beta N\backslash N$, where $\beta N\backslash N$ is considered as a subset of $(l^\infty)^*$. This provides an answer to a question raised by J. Rosenblatt in 1981.

Let $(X,\mathcal{B},P)$ be a nonatomic probability space, G a countable group, and $(g,x) \to gx$ a measure preserving ergodic action of G on $(X,\mathcal{B},P)$. Then G also acts on $L^\infty(X) = L^\infty(X,\mathcal{B},P)$: $(g\cdot f)(x) = f(g^{-1}x)$, $f \in L^\infty(X)$, $g\in G$, $x \in X$. A positive linear functional of norm 1 on $L^\infty(X)$ is called a mean. A mean m is said to be G-invariant if $m(g\cdot f) = m(f)$ for $g \in G$ and $f \in L^\infty(X)$. We will denote the set of G-invariant means on $L^\infty(X)$ by $M(X,G)$. If we consider P as the functional on $L^\infty(X)$ that sends f to $\int f\,dP$, then $P \in M(X,G)$.

It is natural to ask under what conditions will P be the unique G-invariant mean on $L^\infty(X)$. This problem was first studied by del Junco and Rosenblatt [**3**]. They proved that if G is amenable then $M(X,G) \supsetneqq \{P\}$ by showing that X contains small almost invariant sets. The existence of almost invariant sets was also studied by Schmidt [**10**]. These two papers inspired further investigations of G-invariant means by Rosenblatt [**9**], Schmidt [**11**], and Losert and Rindler [**6**], among others.

In this short note we will only address a question raised by Rosenblatt in [**9**]: If G is amenable then $M(X,G) \supsetneqq \{P\}$; what can be said about the cardinality of $M(X,G)$? We will show that whenever $M(X,G)$ is not a singleton (in particular, when G is amenable) then card $M(X,G) \geq 2^c$, where c is the cardinality of the continuum. To prove our result, we will need the following fundamental result of Rosenblatt [**9**, Theorem 1.4]; see also [**3**].

THEOREM A (ROSENBLATT [**9**]). *$M(X,G) \supsetneqq \{P\}$ if and only if there exists a sequence of measurable sets $\{A_n\}$ in X such that*

(i) $P(A_n) > 0$, $\lim_n P(A_n) = 0$,

(ii) *for each* $g \in G$, $\lim_n P(A_n \triangle gA_n)/P(A_n) = 0$.

A sequence $\{A_n\}$ that satisfies (i) and (ii) is called an arbitrarily small asymptotically invariant sequence in [**9**].

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Let $\mathcal{F} = \{\theta \in (l^\infty)^* : \theta \geq 0, \|\theta\| = 1$, and $\theta((t_n)) = 0$ whenever $(t_n) \in l^\infty$ and $\lim_n t_n = 0\}$. Let N be the set of positive integers with discrete topology and βN its Stone-Čech compactification. Then $\beta N \backslash N$ can be considered as a subset of $\mathcal{F}$, and consequently, card $\mathcal{F} = 2^c$. (In fact, the w^*-closed convex hull of $\beta N \backslash N$ is $\mathcal{F}$.) We are now ready to state and prove our main result.

THEOREM. *Let G be a countable group and let $(g, x) \to gx$ be a measure preserving ergodic action of G on a nonatomic probability space $(X, \mathcal{B}, P)$. If $M(X,G) \supsetneqq \{P\}$ then there exists a linear isometry Λ of $(l^\infty)^*$ into $L^\infty(X)^*$ such that $\Lambda(\mathcal{F}) \subset M(X,G)$; in particular,* card $M(X,G) \geq 2^c$.

PROOF. By Theorem A, there exists a sequence of measurable sets $\{A_n\}$ in X which satisfies conditions (i) and (ii). By taking a subsequence, if necessary, we may replace (i) by

(i)′ $$0 < P(A_{n+1}) < \frac{1}{2^n} P(A_n), \qquad n = 1, 2, \ldots .$$

Write $A_n = B_n \cup C_n$ where $B_n = A_n \backslash (A_{n+1} \cup A_{n+2} \cup \cdots)$ and $C_n = A_n \cap (A_{n+1} \cup A_{n+2} \cup \cdots)$. Then, by (i)′,

(iii) $$\begin{aligned} P(C_n) &\leq \sum_{j=n+1}^{\infty} P(A_j) \leq \frac{1}{2^n} P(A_n) + \frac{1}{2^{n+1}} P(A_{n+1}) + \cdots \\ &< \frac{1}{2^n} P(A_n) \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \right) = \frac{1}{2^{n-1}} P(A_n). \end{aligned}$$

If $g \in G$, then $B_n \backslash gB_n \subset (A_n \backslash gA_n) \cup gC_n$, and hence

$$\frac{P(B_n \backslash gB_n)}{P(B_n)} \leq \left(\frac{P(A_n \backslash gA_n)}{P(A_n)} + \frac{P(gC_n)}{P(A_n)} \right) \frac{P(A_n)}{P(B_n)}.$$

Note that by (ii), $P(A_n \backslash gA_n)/P(A_n) \to 0$, and by (iii),

$$P(gC_n)/P(A_n) = P(C_n)/P(A_n) \leq 1/2^{n-1} \to 0$$

and

$$P(A_n)/P(B_n) = P(A_n)/(P(A_n) - P(C_n)) \leq 1/(1 - 1/2^{n-1}) \to 1.$$

So $P(B_n \backslash gB_n)/P(B_n) \to 0$. Therefore, we have constructed a sequence of measurable sets $\{B_n\}$ in X such that

(a) $P(B_n) > 0$, $\lim_n P(B_n) = 0$,

(b) $\lim_n P(B_n \triangle gB_n)/P(B_n) = 0$ for $g \in G$,

(c) $B_n \cap B_k = \varnothing$ if $n \neq k$.

Proceed now as in the proof of Theorem 3.3 of [**2**]. Let $\pi\colon L^\infty(X) \to l^\infty$ be defined by $(\pi f)(n) = (1/P(B_n)) \int_{B_n} f\, dP$. It is easily checked that π is linear, $\|\pi\| = 1$, and $\pi \geq 0$. Given $(t_n) \in l^\infty$, let $f = \sum t_n \chi_{B_n}$. Then, by (c), $\pi f = (t_n)$. Therefore π is onto and hence π^* is an isometry. To see that π^* is the isometry that we are looking for, it remains to show that if $\theta \in \mathcal{F}$ then $\pi^* \theta \in M(X,G)$. Indeed, if $\theta \in \mathcal{F}$, $g \in G$, and $f \in L^\infty(X)$, then

$$\begin{aligned} |\pi(f - g \cdot f)(n)| &= \left| \frac{1}{P(B_n)} \left(\int_{B_n} f\, dP - \int_{B_n} (g \cdot f)\, dP \right) \right| \\ &\leq \|f\|_\infty \frac{P(B_n \triangle gB_n)}{P(B_n)} \to 0, \quad \text{as } n \to \infty, \text{ by (b)}. \end{aligned}$$

Since $\theta \in \mathcal{F}$, $\theta(\pi(f - g \cdot f)) = 0$ or $\pi^*\theta(f) = \pi^*\theta(g \cdot f)$. Thus $\pi^*\theta \in M(X, G)$ and the proof is complete.

COROLLARY. *Suppose that G is a countable amenable group acting ergodically as measure preserving transformations on a nonatomic probability space $(X, \mathcal{B}, P)$. Then* $\operatorname{card} M(X, G) \geq 2^c$.

PROOF. By Theorem 2.1 of [**9**], $\operatorname{card} M(X, G) \geq 2$ and hence by the above theorem, $\operatorname{card} M(X, G) \geq 2^c$.

REMARKS. (1) We have considered similar isometric embeddings of $\mathcal{F}$ in [**1** and **2**]. In particular, we could have quoted Theorem 2.4 of [**2**] to shorten the proof of our theorem here. Since the proof of Theorem 2.4 is quite involved, we prefer the direct construction of $\{B_n\}$ as given above. Recently, Granirer [**5**] has studied embeddings of $\mathcal{F}$ that are not necessarily isometric, in more general settings, by applying H. Rosenthal's fundamental canonical l^1-basis theorem.

(2) It is known that a countable group G has Kazhdan's Property T if and only if $L^\infty(X)$ always has a unique G-invariant mean whenever G acts ergodically as measure preserving transformations on nonatomic probability space $(X, \mathcal{B}, P)$ and G is amenable if and only if no such actions of G admit unique G-invariant means (see Schmidt [**11**]). The group $\mathrm{SL}(2, \mathbf{Z})$ is neither amenable nor does it have property T. Schmidt [**10**] showed that the natural action of $\mathrm{SL}(2, \mathbf{Z})$ on $\mathbf{T}^2$ has no arbitrarily small asymptotically invariant sequences and hence $L^\infty(\mathbf{T}^2)$ has a unique $\mathrm{SL}(2, \mathbf{Z})$-invariant mean (see also [**6** and **9**]). On the other hand, an explicit ergodic action of $\mathrm{SL}(2, \mathbf{Z})$ that admits more than one G-invariant mean is described in [**11**]. By our theorem, this action admits at least 2^c G-invariant means.

(3) Using Theorem A, Margulis [**7**] and Sullivan [**12**] proved independently that, for $n \geq 4$, any $\mathrm{SO}(n+1)$-invariant mean on $L^\infty(S^n, \mathcal{M}, m_n)$ is proportional to the Lebesgue measure m_n. ($\mathcal{M}$ is the σ-algebra of Lebesgue measurable subsets of S^n.) More recently, Drinfel'd [**4**] was able to extend their result to the cases $n = 2$ and 3. Thus the Banach-Ruziewicz problem for S^n has been completely solved. The corresponding Banach-Ruziewicz problem for $\mathbf{R}^n$, $n \geq 3$, was solved by Margulis [**8**].

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