ERGODIC GROUP ACTIONS
WITH NONUNIQUE INVARIANT MEANS
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ABSTRACT. Let $M(X,G)$ be the set of $G$-invariant means on $L^\infty(X,\mathcal{B},P)$, where $G$ is a countable group acting ergodically as measure preserving transformations on a nonatomic probability space $(X,\mathcal{B},P)$. We show that if there exists $\beta \in M(X,G)$, $\beta \neq P$, then $M(X,G)$ contains an isometric copy of $\beta\mathbb{N}/\mathbb{N}$, where $\beta\mathbb{N}/\mathbb{N}$ is considered as a subset of $(L^\infty)^*$. This provides an answer to a question raised by J. Rosenblatt in 1981.

Let $(X,\mathcal{B},P)$ be a nonatomic probability space, $G$ a countable group, and $(g,x) \to gx$ a measure preserving ergodic action of $G$ on $(X,\mathcal{B},P)$. Then $G$ also acts on $L^\infty(X) = L^\infty(X,\mathcal{B},P)$: $(g \cdot f)(x) = f(g^{-1}x)$, $f \in L^\infty(X)$, $g \in G$, $x \in X$. A positive linear functional of norm 1 on $L^\infty(X)$ is called a mean. A mean $m$ is said to be $G$-invariant if $m(g \cdot f) = m(f)$ for $g \in G$ and $f \in L^\infty(X)$. We will denote the set of $G$-invariant means on $L^\infty(X)$ by $M(X,G)$. If we consider $P$ as the functional on $L^\infty(X)$ that sends $f$ to $\int f \, dP$, then $P \in M(X,G)$.

It is natural to ask under what conditions will $P$ be the unique $G$-invariant mean on $L^\infty(X)$. This problem was first studied by del Junco and Rosenblatt [3]. They proved that if $G$ is amenable then $M(X,G) \supseteq \{P\}$ by showing that $X$ contains small almost invariant sets. The existence of almost invariant sets was also studied by Schmidt [10]. These two papers inspired further investigations of $G$-invariant means by Rosenblatt [9], Schmidt [11], and Losert and Rindler [6], among others.

In this short note we will only address a question raised by Rosenblatt in [9]: If $G$ is amenable then $M(X,G) \supseteq \{P\}$; what can be said about the cardinality of $M(X,G)$? We will show that whenever $M(X,G)$ is not a singleton (in particular, when $G$ is amenable) then card $M(X,G) \geq 2^c$, where $c$ is the cardinality of the continuum. To prove our result, we will need the following fundamental result of Rosenblatt [9, Theorem 1.4]; see also [3].

**Theorem A (Rosenblatt [9]).** $M(X,G) \supseteq \{P\}$ if and only if there exists a sequence of measurable sets $\{A_n\}$ in $X$ such that

(i) $P(A_n) > 0$, $\lim_n P(A_n) = 0$,
(ii) for each $g \in G$, $\lim_n P(A_n \triangle gA_n)/P(A_n) = 0$.

A sequence $\{A_n\}$ that satisfies (i) and (ii) is called an arbitrarily small asymptotically invariant sequence in [9].

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Let $\mathcal{F} = \{ \theta \in (l^\infty)^* : \theta \geq 0, \| \theta \| = 1, \text{ and } \theta((t_n)) = 0 \text{ whenever } (t_n) \in l^\infty \text{ and } \lim_n t_n = 0 \}$. Let $N$ be the set of positive integers with discrete topology and $\beta N$ its Stone-Cech compactification. Then $\beta N \setminus N$ can be considered as a subset of $\mathcal{F}$, and consequently, card $\mathcal{F} = 2^\mathbb{C}$. (In fact, the $w^*$-closed convex hull of $\beta N \setminus N$ is $\mathcal{F}$.)

We are now ready to state and prove our main result.

**Theorem.** Let $G$ be a countable group and let $(g, x) \to gx$ be a measure-preserving ergodic action of $G$ on a nonatomic probability space $(X, B, \mu)$. If $M(X, G) \supseteq \{ P \}$ then there exists a linear isometry $A$ of $(l^\infty)^*$ into $L^\infty(X)^*$ such that $A(\mathcal{F}) \subseteq M(X, G)$; in particular, card $M(A, G) > 2^\mathbb{C}$.

**Proof.** By Theorem A, there exists a sequence of measurable sets $\{ A_n \}$ in $X$ which satisfies conditions (i) and (ii). By taking a subsequence, if necessary, we may replace (i) by

\[(i')\quad 0 < P(A_{n+1}) < \frac{1}{2^n} P(A_n), \quad n = 1, 2, \ldots \]

Write $A_n = B_n \cup C_n$ where $B_n = A_n \setminus (A_{n+1} \cup A_{n+2} \cup \cdots)$ and $C_n = A_n \cap (A_{n+1} \cup A_{n+2} \cup \cdots)$. Then, by (i'),

\[P(C_n) \leq \sum_{j=n+1}^\infty P(A_j) \leq \frac{1}{2^n} P(A_n) + \frac{1}{2^{n+1}} P(A_{n+1}) + \cdots \]

\[< \frac{1}{2^n} P(A_n) \left( 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \right) =\frac{1}{2^{n-1}} P(A_n).\]

If $g \in G$, then $B_n \setminus gB_n \subseteq (A_n \setminus gA_n) \cup gC_n$, and hence

\[\frac{P(B_n \setminus gB_n)}{P(B_n)} \leq \left( \frac{P(A_n \setminus gA_n)}{P(A_n)} + \frac{P(gC_n)}{P(A_n)} \right) \frac{P(A_n)}{P(B_n)}.\]

Note that by (ii), $P(A_n \setminus gA_n)/P(A_n) \to 0$, and by (iii),

\[P(gC_n)/P(A_n) = P(C_n)/P(A_n) \leq 1/2^{n-1} \to 0\]

and

\[P(A_n)/P(B_n) = P(A_n)/(P(A_n) - P(C_n)) \leq 1/(1 - 1/2^{n-1}) \to 1.\]

So $P(B_n \setminus gB_n)/P(B_n) \to 0$. Therefore, we have constructed a sequence of measurable sets $\{ B_n \}$ in $X$ such that

(a) $P(B_n) > 0$, lim$_n P(B_n) = 0$,

(b) lim$_n P(B_n \Delta gB_n)/P(B_n) = 0$ for $g \in G$,

(c) $B_n \cap B_k = \emptyset$ if $n \neq k$.

Proceed now as in the proof of Theorem 3.3 of [2]. Let $\pi : L^\infty(X) \to l^\infty$ be defined by $(\pi f)(n) = (1/P(B_n)) \int_{B_n} f dP$. It is easily checked that $\pi$ is linear, $\| \pi \| = 1$, and $\pi \geq 0$. Given $(t_n) \in l^\infty$, let $f = \sum t_n \chi_{B_n}$. Then, by (c), $\pi f = (t_n)$. Therefore $\pi$ is onto and hence $\pi^*$ is an isometry. To see that $\pi^*$ is the isometry that we are looking for, it remains to show that if $\theta \in \mathcal{F}$ then $\pi^* \theta \in M(X, G)$. Indeed, if $\theta \in \mathcal{F}$, $g \in G$, and $f \in L^\infty(X)$, then

\[|\pi(f - g \cdot f)(n)| = \left| \frac{1}{P(B_n)} \left( \int_{B_n} f dP - \int_{B_n} (g \cdot f) dP \right) \right| \leq \|f\|_{L^\infty} \frac{P(B_n \Delta gB_n)}{P(B_n)} \to 0, \quad \text{as } n \to \infty, \quad \text{by (b)}.\]
Since \( \theta \in \mathcal{F}, \theta(\pi(f - g \cdot f)) = 0 \) or \( \pi^* \theta(f) = \pi^* \theta(g \cdot f) \). Thus \( \pi^* \theta \in M(X, G) \) and the proof is complete.

**Corollary.** Suppose that \( G \) is a countable amenable group acting ergodically as measure preserving transformations on a nonatomic probability space \((X, \mathcal{B}, P)\). Then \( \text{card} \ M(X, G) \geq 2^c \).

**Proof.** By Theorem 2.1 of [9], \( \text{card} \ M(X, G) \geq 2 \) and hence by the above theorem, \( \text{card} \ M(X, G) \geq 2^c \).

**Remarks.** (1) We have considered similar isometric embeddings of \( \mathcal{F} \) in [1 and 2]. In particular, we could have quoted Theorem 2.4 of [2] to shorten the proof of our theorem here. Since the proof of Theorem 2.4 is quite involved, we prefer the direct construction of \( \{B_n\} \) as given above. Recently, Granirer [5] has studied embeddings of \( \mathcal{F} \) that are not necessarily isometric, in more general settings, by applying H. Rosenthal's fundamental canonical \( l^1 \)-basis theorem.

(2) It is known that a countable group \( G \) has Kazhdan's Property T if and only if \( L^\infty(X) \) always has a unique \( G \)-invariant mean whenever \( G \) acts ergodically as measure preserving transformations on nonatomic probability space \((X, \mathcal{B}, P)\) and \( G \) is amenable if and only if no such actions of \( G \) admit unique \( G \)-invariant means (see Schmidt [11]). The group \( SL(2,\mathbb{Z}) \) is neither amenable nor does it have property T. Schmidt [10] showed that the natural action of \( SL(2,\mathbb{Z}) \) on \( T^2 \) has no arbitrarily small asymptotically invariant sequences and hence \( L^\infty(T^2) \) has a unique \( SL(2,\mathbb{Z}) \)-invariant mean (see also [6 and 9]). On the other hand, an explicit ergodic action of \( SL(2,\mathbb{Z}) \) that admits more than one \( G \)-invariant mean is described in [11]. By our theorem, this action admits at least \( 2^c \) \( G \)-invariant means.

(3) Using Theorem A, Margulis [7] and Sullivan [12] proved independently that, for \( n \geq 4 \), any \( SO(n + 1) \)-invariant mean on \( L^\infty(S^n, M, m_n) \) is proportional to the Lebesgue measure \( m_n \). (\( M \) is the \( \sigma \)-algebra of Lebesgue measurable subsets of \( S^n \)). More recently, Drinfel'd [4] was able to extend their result to the cases \( n = 2 \) and 3. Thus the Banach-Ruziewicz problem for \( S^n \) has been completely solved. The corresponding Banach-Ruziewicz problem for \( \mathbb{R}^n, n \geq 3 \), was solved by Margulis [8].

**References**


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