ON THE DILATATION ESTIMATES FOR BEURLING-AHLFORS QUASICONFORMAL EXTENSION

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ABSTRACT. Let $\mu(x)$ be a $\rho$-quasisymmetric function. Then the dilatation $K(z)$ of Beurling-Ahlfors extension with $r = 1$ satisfies the inequalities $K \leq 2\rho - 7(\rho - 1)/(\rho + 1)$ and $K < 2\rho - 2 + O(1/\rho)$ for sufficiently large $\rho$.

1. Introduction. Let $\mu(x)$ be $\rho$-quasisymmetric, $1 \leq \rho < \infty$, i.e. $\mu(x)$ is a continuous increasing function mapping the real line onto itself and satisfying

\begin{equation}
\frac{1}{\rho} \leq \frac{\mu(x + t) - \mu(x)}{\mu(x) - \mu(x - t)} \leq \rho
\end{equation}

for all $x$ and $t \neq 0$.

Beurling and Ahlfors [1] using the formulas

\begin{equation}
\begin{align*}
 u(x, y) &= \frac{1}{2} \int_0^1 [\mu(x + ty) - \mu(x - ty)] dt, \\
 v(x, y) &= \frac{r}{2} \int_0^1 [\mu(x + ty) - \mu(x - ty)] dt \quad (r > 0)
\end{align*}
\end{equation}

constructed the function $w(z) = u(x, y) + iv(x, y)$, which is a q.c. mapping from the upper half-plane onto itself having $\mu(x)$ as its boundary value. It is of interest to estimate the dilatation $K(z)$ of $w(z)$.

Beurling and Ahlfors [1] first proved for some $r$

\begin{equation}
K \leq \rho^2.
\end{equation}

Reed [2] improved the inequality (3) as follows:

\begin{equation}
K < 8\rho.
\end{equation}

Li Zhong [3] then obtained

\begin{equation}
K < 4 \cdot 2\rho.
\end{equation}

Ahlfors [4] also proved that the Beurling-Ahlfors extension function with $r = 1$ is quasi-isometric, i.e. there exists a constant $A$ such that

\begin{equation}
\frac{1}{A} d(z_1, z_2) \leq d(w(z_1), w(z_2)) \leq A d(z_1, z_2)
\end{equation}

for any $z_1, z_2$ in the upper half-plane, where $d(\cdot, \cdot)$ denotes the non-Euclidean distance.

Ahlfors obtained

\begin{equation}
A < 4\rho^2(\rho + 1).
\end{equation}
Chen Ji-xiu refined Reed’s method to obtain

(8) \[ K < 2 \cdot 58 \rho \]

and

(9) \[ A < 3 \rho. \]

In this paper we have refined the Beurling-Ahlfors technique and obtained the following theorem.

**THEOREM 1.** Let \( \mu(x) \) be a \( \rho \)-quasisymmetric function. Then the dilatation \( K(z) \) of the Beurling-Ahlfors extension with \( r = 1 \) satisfies the inequalities

(10) \[ K \leq 2 \rho - \frac{7(\rho - 1)}{6(\rho + 1)} \]

and

(11) \[ K < 2 \rho - 2 + O(1/\rho) \]

for sufficiently large \( \rho \).

**THEOREM 2.** Let \( \mu(x) \) be a \( \rho \)-quasisymmetric function and let \( w(z) \) be the Beurling-Ahlfors extension of \( \mu(x) \) for \( r = 1 \). Then

(12) \[ \frac{1}{2\rho} d(z_1, z_2) \leq d(w(z_1), w(z_2)) \leq 2\rho d(z_1, z_2) \]

for any \( z_1, z_2 \) in the upper half-plane.

**Remark 1.** When \( \rho = 1 \) and \( \mu(0) = 0 \), we easily see that \( w(z) = c(x + yi/2) \) (\( c > 0 \)). It is evident that \( K(z) = 2 \) and

\[
\lim_{\text{Im} z_1 = \text{Im} z_2 > 0} \frac{d(w(z_1), w(z_2))}{d(z_1, z_2)} = 2.
\]

Therefore the coefficient 2 of \( \rho \) either in Theorem 1 or 2 cannot be replaced by any smaller number.

**Remark 2.** This work was done during the summer of 1983. It is independent of Lehtinen’s paper. Lehtinen [5] obtained

(13) \[ K \leq 2 \rho. \]

2. **Lemma.** Let \( \mu(x) \) be a \( \rho \)-quasisymmetric function normalized by \( \mu(0) = 0 \) and \( \mu(1) = 1 \). Then

(14) \[ (1 + 2\rho) \xi + \beta \eta \geq 1 + \beta, \]

(15) \[ \beta(1 + 2\rho) \eta + \xi \geq 1 + \beta, \]

where \( \beta = -\mu(-1), \xi = 1 - \int_0^1 \mu(t) \, dt \), and \( \eta = 1 + \beta^{-1} \int_{-1}^0 \mu(t) \, dt \).

**Proof.** Taking \( x = t > 0 \) in (1) we have \( (1 + \rho)\mu(x) \geq \mu(2x) \). Thus

\[
(1 + \rho) \int_0^1 \mu(x) \, dx \geq \int_0^1 \mu(2x) \, dx = \frac{1}{2} \int_0^1 \mu(x) \, dx + \frac{1}{2} \int_0^2 \mu(x) \, dx.
\]
Therefore

\[(1 + 2\rho) \int_0^1 \mu(x) \, dx \geq \int_1^2 \mu(x) \, dx.\]

Substituting \(1 - \mu(1 - x)\) for \(\mu(x)\), we get

\[(1 + 2\rho) \left[ 1 - \int_0^1 \mu(x) \, dx \right] \geq 1 - \int_{-1}^0 \mu(x) \, dx.\]

This yields (14). Similarly, substituting \(1 + \mu(x - 1)/\beta\) for \(\mu(x)\) yields (15).

3. **Proof of Theorem 1.** Because of linear invariance we only need to estimate \(K(z)\) for \(x = 0, y = 1, \) and \(\mu(x)\) normalized by \(\mu(0) = 0, \mu(1) = 1\). Thus the dilatation \(K = K(t)\) with \(r = 1\) satisfies \([1]\) the equation

\[(17) \quad K + \frac{1}{K} = \frac{1}{\xi + \eta} \left[ \frac{1}{\beta} (1 + \xi^2) + \beta (1 + \eta^2) \right] \equiv F(\xi, \eta, \beta),\]

where \(\beta \leq \rho, 1/(1 + \rho) \leq \xi, \eta \leq \rho/(1 + \rho)\). Furthermore, we can suppose \(\beta \geq 1\), otherwise consider \(-w(-z)/\beta\).

Let \(\mathcal{C}\) be a closed domain bounded by a polygon \(ABCD\). The side \(AB\) lies on the line of \((1 + 2\rho)\xi + \beta \eta = 1 + \beta\); the other sides \(BC, CD, DE,\) and \(AE\) lie on the lines of \(\xi = 1/(1 + \rho), \eta = \rho/(1 + \rho), \xi = \rho/(1 + \rho),\) and \(\eta = 1/(1 + \rho)\), respectively. It is sufficient to look at the maximum of \(F(\xi, \eta, \beta)\) in \(\mathcal{C}\).

By calculating we have

\[
\frac{\partial F}{\partial \eta} = \frac{\beta(\xi + \eta)^2 - (\beta + 1/\beta)(1 + \xi^2)}{(\xi + \eta)^2}, \quad \frac{\partial^2 F}{\partial \eta^2} = \frac{2(\beta + 1/\beta)(1 + \xi^2)}{(\xi + \eta)^3}.
\]

Since \(\partial^2 F/\partial \eta^2 > 0\), max of \(F\) in \(\mathcal{C}\) is in \(CD \cup AE \cup AB\). Since \(\partial^2 F/\partial \xi^2 > 0\), the max is in \(DE \cup BC \cup AB\), so the max is in \(AB \cup \{C, D, E\}\). Since \(\partial F/\partial \eta < 0\) in \(BC\) and \(\partial F/\partial \xi < 0\) in \(AE\), the max is in \(AB \cup \{D\}\).

When \((\xi, \eta) \in AB\), then \(\beta \eta = 1 + \beta - (1 + 2\rho)\xi\) and

\[(18) \quad F(\xi, \eta, \beta) = \frac{2(1 + \beta + \beta^2 - (1 + \beta)(1 + 2\rho)\xi + (1 + 2\rho + 2\rho^2)\xi^2}{1 + \beta - (1 + 2\rho - \beta)\xi} \equiv 2W(\xi, \beta).\]
But $W(\xi, \beta)$ is a convex function of either $\beta$ or $\xi$, therefore the max of $W(\xi, \beta)$ must occur when $\xi = 1/(1 + \rho)$, $\beta = \rho$, or

$$\xi = \frac{1 + \rho + \rho \beta}{(1 + \rho)(1 + 2\rho)}, \quad 1 \leq \beta \leq \rho.$$ 

Hence we only need to consider the following cases:

**Case 1:** At point $D$. Then $\xi = \eta = \rho/(1 + \rho)$,

$$F\left(\frac{\rho}{1 + \rho}, \frac{\rho}{1 + \rho}, \beta\right) = \left(\beta + \frac{1}{\beta}\right) \cdot \frac{1 + (\rho/(1 + \rho)^2)}{2\rho/(1 + \rho)} \leq 2\rho + \frac{1}{2\rho} - \frac{(\rho - 1)(2\rho^3 + 4\rho^2 + 2\rho + 1)}{2\rho^2(\rho + 1)}.$$ 

**Case 2:** At point $B$ with $\beta = \rho$. Then $\xi = 1/(1 + \rho)$, $\eta = \rho/(1 + \rho)$,

$$F\left(\frac{1}{1 + \rho}, \frac{\rho}{1 + \rho}, \rho\right) = 2\rho - 2 + \frac{2}{\rho} + \frac{2}{1 + \rho} - \frac{2}{(1 + \rho)^2} = 2\rho + \frac{1}{2\rho} - \frac{(\rho - 1)(4\rho^2 + 5\rho + 3)}{2\rho(\rho + 1)}.$$ 

**Case 3:** At point $A$. Then $\xi = (1 + \rho + \beta\rho)/(1 + \rho)(1 + 2\rho)$, $1 \leq \beta \leq \rho$, $\eta = 1/(1 + \rho)$,

$$F\left(\frac{1 + \rho + \beta\rho}{(1 + \rho)(1 + 2\rho)}, \frac{1}{1 + \rho}, \beta\right) \leq (1 + \rho)(1 + 2\rho) \cdot \frac{\beta(1 + \beta) + (1 + \rho) + \beta^2/(1 + \rho) + [(1 + \rho + \beta\rho)^2/(1 + \rho)(1 + 2\rho)]^2/\beta}{2 + 3\rho + \beta\rho}.$$ 

Denote $\lambda(\rho) = [(1 + \rho + \rho^2)/(1 + \rho)(1 + 2\rho)]^2$. Then

$$(2 + 3\rho + \beta\rho)^2 \frac{\partial Y}{\partial \beta} = (2 + 3\rho) \left[1 + \frac{1}{(1 + \rho)^2}\right] - [1 + \lambda(\rho)] \left[\frac{2\rho}{\beta} + \frac{1}{\beta^2} (2 + 3\rho)\right].$$

When $\beta$ increases from 1 to $\rho$, the sign of $\partial Y/\partial \beta$ changes only once. Hence

$$\max_{1 \leq \beta \leq \rho} Y(\beta, \rho) = \max\{Y(1, \rho), Y(\rho, \rho)\},$$

and

$$\begin{align*}
(1 + \rho)(1 + 2\rho)Y(1, \rho) &= \frac{9}{8}\rho + 1 + \frac{1}{1 + \rho} - \frac{9}{16(1 + 2\rho)} - \frac{9}{16(1 + 2\rho)^2} \\
&= 2\rho + \frac{1}{2\rho} - \frac{(\rho - 1)(7\rho^4 + 13\rho^3 + 4\rho^2 - 2\rho - 1)}{2\rho(1 + \rho)(1 + 2\rho)^2},
\end{align*}$$

$$\begin{align*}
(1 + \rho)(1 + 2\rho)Y(\rho, \rho) &= 2\rho - 3 + \frac{1}{\rho} + \frac{15}{2 + \rho} - \frac{3}{1 + 2\rho} - \frac{4}{1 + \rho} + \frac{2}{(1 + \rho)^2} \\
&= 2\rho + \frac{1}{2\rho} - \frac{(\rho - 1)(12\rho^4 + 26\rho^3 + 23\rho^2 + 9\rho + 2)}{2\rho(\rho + 1)^2(\rho + 2)(2\rho + 1)}.
\end{align*}$$
From (19), (20), (22), and (23) we have

\begin{equation}
K + \frac{1}{K} \leq 2\rho + \frac{1}{2\rho} - \frac{7(\rho - 1)}{6(\rho + 1)}
\end{equation}

and

\begin{equation}
K + \frac{1}{K} \leq 2\rho + \frac{1}{2\rho} - 2 + O\left(\frac{1}{\rho}\right)
\end{equation}

for sufficiently large \( \rho \). Inequalities (10) and (11) follow.

4. Proof of Theorem 2. Because the non-Euclidean distance is also a linear invariant we only need to prove

\begin{equation}
\frac{1}{2\rho} \leq \left| \frac{dw(i)}{v(i)dz} \right| \leq 2\rho
\end{equation}

for \( \mu(0) = 0 \) and \( \mu(1) = 1 \). Similarly, we suppose \( \beta \geq 1 \). From (2)

\[ v(i) = \frac{1}{2} \int_0^1 [\mu(t) - \mu(-t)] dt = \frac{1}{2}(1 + \beta) - \frac{1}{2}(\xi + \eta). \]

Then

\[ \frac{1 + \beta}{2(1 + \rho)} \leq v(i) \leq \frac{\rho(1 + \beta)}{2(1 + \rho)}. \]

From [4]

\[ |w_z(i)|^2 = \frac{1}{8}[(1 + \xi^2) + \beta^2(1 + \eta^2) + 2\beta(\xi + \eta)]. \]

Then

\[ |w_z(i)|^2 \leq \frac{1}{8} \left[ 1 + \frac{\rho^2}{(1 + \rho)^2} + \beta \left( 1 + \frac{\rho^2}{(1 + \rho)^2} \right) + \frac{4\beta\rho}{1 + \rho} \right] = \left( \frac{1 + \beta}{2(1 + \rho)^2} \right)^2. \]

Hence

\begin{equation}
\left| \frac{w_z(i)}{v(i)} \right|^2 \leq \frac{(1 + \beta)^2(2\rho + 2\rho + 1) - 2\beta}{8(1 + \rho)^2} \cdot \frac{4(1 + \rho)^2}{(1 + \beta)^2}
\end{equation}

\begin{equation}
= \frac{1}{2} \left[ 2\rho^2 + 2\rho + 1 - \frac{2\beta}{(1 + \beta)^2} \right] \leq \frac{1}{2} \left[ 2\rho^2 + 2\rho + 1 - \frac{2\rho}{(1 + \rho)^2} \right],
\end{equation}

\begin{equation}
\left| \frac{w_z(i)}{v(i)} \right|^2 \geq \frac{1}{8} \frac{4(1 + \rho)^2}{\rho^2(1 + \beta)^2} \geq \frac{(1 + \rho)^2}{4\rho^2}.
\end{equation}

From Theorem 1

\[ K \leq 2\rho - \frac{\rho - 1}{\rho + 1} = \frac{2\rho^2 + \rho + 1}{\rho + 1}. \]

Then

\begin{equation}
\frac{K}{K + 1} \leq \frac{2\rho^2 + \rho + 1}{2\rho^2 + 2\rho + 2}.
\end{equation}
For $dw = w_z \, dz + w_z \, d\bar{z}$, we have

$$|dw| \leq |w_z| \left( 1 + \left| \frac{w_z}{w_z} \right| \right) |dz| = \frac{2K}{K + 1} |w_z \, dz|,$$

$$|dw| \geq |w_z| \left( 1 - \left| \frac{w_z}{w_z} \right| \right) |dz| = \frac{2}{K + 1} |w_z \, dz|.$$  

Then

$$\left| \frac{dw(i)}{V(i) \, dz} \right|^2 \leq \left( \frac{4K^2}{(K + 1)^2} \cdot \left| \frac{w_z(i)}{v(i)} \right| \right)^2$$

$$\leq 2 \left( \frac{2 \rho^2 + \rho + 1}{2 \rho^2 + 2 \rho + 2} \right)^2 \left[ 2\rho^2 + 2\rho + 1 - \frac{2\rho}{(1 + \rho)^2} \right]$$

$$= 4\rho^2 - \frac{2(\rho - 1)(2\rho^5 + 12\rho^4 + 15\rho^3 + 13\rho^2 + 5\rho + 1)}{2(1 + \rho)^2(1 + \rho + \rho^2)^2}$$

$$\leq 4\rho^2,$$

and that completes the proof.

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**REFERENCES**


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