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ON THE DILATATION ESTIMATES FOR BEURLING-AHLFORS QUASICONFORMAL EXTENSION

DELIN TAN

ABSTRACT. Let $\mu(x)$ be a ρ-quasisymmetric function. Then the dilatation $K(z)$ of Beurling-Ahlfors extension with $r = 1$ satisfies the inequalitites $K \leq 2\rho - 7(\rho - 1)/6(\rho + 1)$ and $K < 2\rho - 2 + O(1/\rho)$ for sufficiently large ρ.

1. Introduction. Let $\mu(x)$ be ρ-quasisymmetric, $1 \leq \rho < \infty$, i.e. $\mu(x)$ is a continuous incresasing function mapping the real line onto itself and satisfying

$$\frac{1}{\rho} \leq \frac{\mu(x+t) - \mu(x)}{\mu(x) - \mu(x-t)} \leq \rho \tag{1}$$

for all x and $t \neq 0$.

Beurling and Ahlfors [**1**] using the formulas

$$\begin{aligned} u(x,y) &= \frac{1}{2}\int_0^1 [\mu(x+ty) - \mu(x-ty)]\,dt, \\ v(x,y) &= \frac{r}{2}\int_0^1 [\mu(x+ty) - \mu(x-ty)]\,dt \qquad (r > 0) \end{aligned} \tag{2}$$

constructed the function $w(z) = u(x,y) + iv(x,y)$ $(z = x + iy)$, which is a q.c. mapping from the upper half-plane onto itself having $\mu(x)$ as its boundary value. It is of interest to estimate the dilatation $K(z)$ of $w(z)$.

Beurling and Ahlfors [**1**] first proved for some r

$$K \leq \rho^2. \tag{3}$$

Reed [**2**] improved the inequality (3) as follows:

$$K < 8\rho. \tag{4}$$

Li Zhong [**3**] then obtained

$$K < 4 \cdot 2\rho. \tag{5}$$

Ahlfors [**4**] also proved that the Beurling-Ahlfors extension function with $r = 1$ is quasi-isometric, i.e. there exists a constant A such that

$$\frac{1}{A}\, d(z_1, z_2) \leq d(w(z_1), w(z_2)) \leq A d(z_1, z_2) \tag{6}$$

for any z_1, z_2 in the upper half-plane, where $d(\cdot,\cdot)$ denotes the non-Euclidean distance.

Ahlfors obtained

$$A < 4\rho^2(\rho + 1). \tag{7}$$

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Chen Ji-xiu refined Reed's method to obtain

$$K < 2 \cdot 58\rho \tag{8}$$

and

$$A < 3\rho. \tag{9}$$

In this paper we have refined the Beurling-Ahlfors technique and obtained the following theorem.

THEOREM 1. *Let $\mu(x)$ be a ρ-quasisymmetric function. Then the dilatation $K(z)$ of the Beurling-Ahlfors extension with $r = 1$ satisfies the inequalities*

$$K \leq 2\rho - \frac{7(\rho - 1)}{6(\rho + 1)} \tag{10}$$

and

$$K < 2\rho - 2 + O(1/\rho) \tag{11}$$

for sufficiently large ρ.

THEOREM 2. *Let $\mu(x)$ be a ρ-quasisymmetric function and let $w(z)$ be the Beurling-Ahlfors extension of $\mu(x)$ for $r = 1$. Then*

$$\frac{1}{2\rho}\, d(z_1, z_2) \leq d(w(z_1), w(z_2)) \leq 2\rho\, d(z_1, z_2) \tag{12}$$

for any z_1, z_2 in the upper half-plane.

REMARK 1. When $\rho = 1$ and $\mu(0) = 0$, we easily see that $w(z) = c(x+yi/2)$ ($c > 0$). It is evident that $K(z) = 2$ and

$$\lim_{\substack{z_1 \to z_2 \\ \operatorname{Im} z_1 = \operatorname{Im} z_2 > 0}} \frac{d(w(z_1), w(z_2))}{d(z_1, z_2)} = 2.$$

Therefore the coefficient 2 of ρ either in Theorem 1 or 2 cannot be replaced by any smaller number.

REMARK 2. This work was done during the summer of 1983. It is independent of Lehtinen's paper. Lehtinen [**5**] obtained

$$K \leq 2\rho. \tag{13}$$

2. LEMMA. *Let $\mu(x)$ be a ρ-quasisymmetric function normalized by $\mu(0) = 0$ and $\mu(1) = 1$. Then*

$$(1 + 2\rho)\xi + \beta\eta \geq 1 + \beta, \tag{14}$$

$$\beta(1 + 2\rho)\eta + \xi \geq 1 + \beta, \tag{15}$$

where $\beta = -\mu(-1)$, $\xi = 1 - \int_0^1 \mu(t)\, dt$, and $\eta = 1 + \beta^{-1} \int_{-1}^0 \mu(t)\, dt$.

PROOF. Taking $x = t > 0$ in (1) we have $(1 + \rho)\mu(x) \geq \mu(2x)$. Thus

$$(1 + \rho) \int_0^1 \mu(x)\, dx \geq \int_0^1 \mu(2x)\, dx = \frac{1}{2} \int_0^1 \mu(x)\, dx + \frac{1}{2} \int_0^2 \mu(x)\, dx.$$

Therefore

$$(1+2\rho)\int_0^1 \mu(x)\,dx \geq \int_1^2 \mu(x)\,dx. \tag{16}$$

Substituting $1-\mu(1-x)$ for $\mu(x)$, we get

$$(1+2\rho)\left[1-\int_0^1 \mu(x)\,dx\right] \geq 1-\int_{-1}^0 \mu(x)\,dx.$$

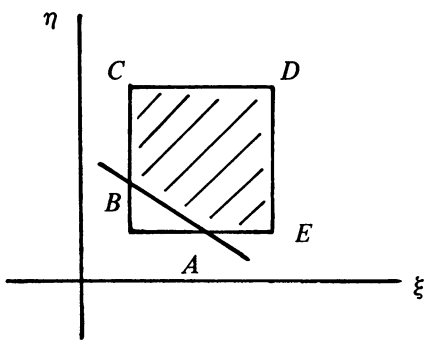
This yields (14). Similarly, substituting $1+\mu(x-1)/\beta$ for $\mu(x)$ yields (15).

3. Proof of Theorem 1. Because of linear invariance we only need to estimate $K(z)$ for $x=0$, $y=1$, and $\mu(x)$ normalized by $\mu(0)=0$, $\mu(1)=1$. Thus the dilatation $K=K(i)$ with $r=1$ satisfies [**1**] the equation

$$K+\frac{1}{K}=\frac{1}{\xi+\eta}\left[\frac{1}{\beta}(1+\xi^2)+\beta(1+\eta^2)\right]\equiv F(\xi,\eta,\beta), \tag{17}$$

where $\beta\leq\rho$, $1/(1+\rho)\leq\xi,\eta\leq\rho/(1+\rho)$. Furthermore, we can suppose $\beta\geq 1$, otherwise consider $-w(-\bar{z})/\beta$.

Let $\overline{G}$ be a closed domain bounded by a polygon $ABCDE$. The side AB lies on the line of $(1+2\rho)\xi+\beta\eta=1+\beta$; the other sides BC, CD, DE, and AE lie on the lines of $\xi=1/(1+\rho)$, $\eta=\rho/(1+\rho)$, $\xi=\rho/(1+\rho)$, and $\eta=1/(1+\rho)$, respectively. It is sufficient to look at the maximum of $F(\xi,\eta,\beta)$ in $\overline{G}$.



By calculating we have

$$\frac{\partial F}{\partial\eta}=\frac{\beta(\xi+\eta)^2-(\beta+1/\beta)(1+\xi^2)}{(\xi+\eta)^2},\quad \frac{\partial^2 F}{\partial\eta^2}=\frac{2(\beta+1/\beta)(1+\xi^2)}{(\xi+\eta)^3}.$$

Since $\partial^2F/\partial\eta^2>0$, max of F in $\overline{G}$ is in $CD\cup AE\cup AB$. Since $\partial^2F/\partial\xi^2>0$, the max is in $DE\cup BC\cup AB$, so the max is in $AB\cup\{C,D,E\}$. Since $\partial F/\partial\eta<0$ in BC and $\partial F/\partial\xi<0$ in AE, the max is in $AB\cup\{D\}$.

When $(\xi,\eta)\in AB$, then $\beta\eta=1+\beta-(1+2\rho)\xi$ and

$$\begin{aligned} F(\xi,\eta,\beta) &= 2\frac{1+\beta+\beta^2-(1+\beta)(1+2\rho)\xi+(1+2\rho+2\rho^2)\xi^2}{1+\beta-(1+2\rho-\beta)\xi} \\ &\equiv 2W(\xi,\beta). \end{aligned} \tag{18}$$

But $W(\xi,\beta)$ is a convex function of either β or ξ, therefore the max of $W(\xi,\beta)$ must occur when $\xi = 1/(1+\rho)$, $\beta = \rho$, or

$$\xi = \frac{1+\rho+\rho\beta}{(1+\rho)(1+2\rho)}, \qquad 1 \le \beta \le \rho.$$

Hence we only need to consider the following cases:

Case 1: *At point D.* Then $\xi = \eta = \rho/(1+\rho)$,

$$\begin{aligned} F\left(\frac{\rho}{1+\rho}, \frac{\rho}{1+\rho}, \beta\right) &= \left(\beta + \frac{1}{\beta}\right) \cdot \frac{1+(\rho/1+\rho)^2}{2\rho/(1+\rho)} \\ &\le 2\rho + \frac{1}{2\rho} - \frac{(\rho-1)(2\rho^3+4\rho^2+2\rho+1)}{2\rho^2(\rho+1)}. \end{aligned} \tag{19}$$

Case 2: *At point B with* $\beta = \rho$. Then $\xi = 1/(1+\rho), \eta = \rho/(1+\rho)$,

$$\begin{aligned} F\left(\frac{1}{1+\rho}, \frac{\rho}{1+\rho}, \rho\right) &= 2\rho - 2 + \frac{2}{\rho} + \frac{2}{1+\rho} - \frac{2}{(1+\rho)^2} \\ &= 2\rho + \frac{1}{2\rho} - \frac{(\rho-1)(4\rho^2+5\rho+3)}{2\rho(\rho+1)^2}. \end{aligned} \tag{20}$$

Case 3: *At point A.* Then $\xi = (1+\rho+\beta\rho)/(1+\rho)(1+2\rho)$, $1 \le \beta \le \rho$, $\eta = 1/(1+\rho)$,

$$\begin{aligned} &F\left(\frac{1+\rho+\beta\rho}{(1+\rho)(1+2\rho)}, \frac{1}{1+\rho}, \beta\right) \\ &\quad = \frac{\beta + 1/\beta + \beta/(1+\rho)^2 + [(1+\rho+\beta\rho)^2/(1+\rho)(1+2\rho)]^2/\beta}{(2+3\rho+\beta\rho)/(1+2\rho)(1+\rho)} \\ &\quad \le (1+\rho)(1+2\rho) \cdot \frac{\beta + 1/\beta + \beta/(1+\rho)^2 + [(1+\rho+\rho^2)/(1+\rho)(1+2\rho)]^2/\beta}{2+3\rho+\beta\rho} \\ &\quad \equiv (1+\rho)(1+2\rho)Y(\beta,\rho). \end{aligned} \tag{21}$$

Denote $\lambda(\rho) = [(1+\rho+\rho^2)/(1+\rho)(1+2\rho)]^2$. Then

$$(2+3\rho+\beta\rho)^2 \frac{\partial Y}{\partial \beta} = (2+3\rho)\left[1 + \frac{1}{(1+\rho)^2}\right] - [1+\lambda(\rho)]\left[\frac{2\rho}{\beta} + \frac{1}{\beta^2}(2+3\rho)\right].$$

When β increases from 1 to ρ, the sign of $\partial Y/\partial\beta$ changes only once. Hence

$$\max_{1\le\beta\le\rho} Y(\beta,\rho) = \max\{Y(1,\rho), Y(\rho,\rho)\},$$

and

$$\begin{aligned} (1+\rho)(1+2\rho)Y(1,\rho) &= \frac{9}{8}\rho + 1 + \frac{1}{1+\rho} - \frac{9}{16(1+2\rho)} - \frac{9}{16(1+2\rho)^2} \\ &= 2\rho + \frac{1}{2\rho} - \frac{(\rho-1)(7\rho^4+13\rho^3+4\rho^2-2\rho-1)}{2\rho(1+\rho)(1+2\rho)^2}, \end{aligned} \tag{22}$$

$$\begin{aligned} (1+\rho)(1+2\rho)Y(\rho,\rho) &= 2\rho - 3 + \frac{1}{\rho} + \frac{15}{2+\rho} - \frac{3}{1+2\rho} - \frac{4}{1+\rho} + \frac{2}{(1+\rho)^2} \\ &= 2\rho + \frac{1}{2\rho} - \frac{(\rho-1)(12\rho^4+26\rho^3+23\rho^2+9\rho+2)}{2\rho(\rho+1)^2(\rho+2)(2\rho+1)}. \end{aligned} \tag{23}$$

From (19), (20), (22), and (23) we have

$$K + \frac{1}{K} \leq 2\rho + \frac{1}{2\rho} - \frac{7(\rho - 1)}{6(\rho + 1)} \tag{24}$$

and

$$K + \frac{1}{K} \leq 2\rho + \frac{1}{2\rho} - 2 + O\left(\frac{1}{\rho}\right) \tag{25}$$

for sufficiently large ρ. Inequalities (10) and (11) follow.

4. Proof of Theorem 2. Because the non-Euclidean distance is also a linear invariant we only need to prove

$$\frac{1}{2\rho} \leq \left|\frac{dw(i)}{v(i)\,dz}\right| \leq 2\rho \tag{26}$$

for $\mu(0) = 0$ and $\mu(1) = 1$. Similarly, we suppose $\beta \geq 1$. From (2)

$$v(i) = \frac{1}{2}\int_0^1 [\mu(t) - \mu(-t)]\,dt = \frac{1}{2}(1 + \beta) - \frac{1}{2}(\xi + \beta\eta).$$

Then

$$\frac{1 + \beta}{2(1 + \rho)} \leq v(i) \leq \frac{\rho(1 + \beta)}{2(1 + \rho)}.$$

From [4]

$$|w_z(i)|^2 = \tfrac{1}{8}[(1 + \xi^2) + \beta^2(1 + \eta^2) + 2\beta(\xi + \eta)].$$

Then

$$\begin{aligned} |w_z(i)^2 &\leq \tfrac{1}{8}\left[1 + \frac{\rho^2}{(1 + \rho)^2} + \beta\left(1 + \frac{\rho^2}{(1 + \rho)^2}\right) + \frac{4\beta\rho}{1 + \rho}\right] \\ &= \frac{(1 + \beta)^2(2\rho^2 + 2\rho + 1) - 2\beta}{8(1 + \rho)^2}, \end{aligned}$$

$$|w_z(i)|^2 \geq \tfrac{1}{8}(1 + \beta^2).$$

Hence

$$\begin{aligned} \left|\frac{w_z(i)}{v(i)}\right|^2 &\leq \frac{(1 + \beta)^2(2\rho + 2\rho + 1) - 2\beta}{8(1 + \rho)^2} \cdot \frac{4(1 + \rho)^2}{(1 + \beta)^2} \\ &= \frac{1}{2}\left[2\rho^2 + 2\rho + 1 - \frac{2\beta}{(1 + \beta)^2}\right] \\ &\leq \frac{1}{2}\left[2\rho^2 + 2\rho + 1 - \frac{2\rho}{(1 + \rho)^2}\right], \end{aligned} \tag{27}$$

$$\left|\frac{w_z(i)}{v(i)}\right|^2 \geq \frac{1}{8}(1 + \beta^2) \cdot \frac{4(1 + \rho)^2}{\rho^2(1 + \beta)^2} \geq \frac{(1 + \rho)^2}{4\rho^2}. \tag{28}$$

From Theorem 1

$$K \leq 2\rho - \frac{\rho - 1}{\rho + 1} = \frac{2\rho^2 + \rho + 1}{\rho + 1}.$$

Then

$$\frac{K}{K + 1} \leq \frac{2\rho^2 + \rho + 1}{2\rho^2 + 2\rho + 2}. \tag{29}$$

For $dw = w_z\,dz + w_{\bar z}\,d\bar z$, we have

$$|dw| \le |w_z|\left(1 + \left|\frac{w_{\bar z}}{w_z}\right|\right)|dz| = \frac{2K}{K+1}|w_z\,dz|,$$
$$|dw| \ge |w_z|\left(1 - \left|\frac{w_{\bar z}}{w_z}\right|\right)|dz| = \frac{2}{K+1}|w_z\,dz|.$$

Then

$$\begin{aligned}\left|\frac{dw(i)}{V(i)\,dz}\right|^2 &\le \frac{4K^2}{(K+1)^2}\cdot\left|\frac{w_z(i)}{v(i)}\right|^2\\ &\le 2\left(\frac{2\rho^2+\rho+1}{2\rho^2+2\rho+2}\right)^2\left[2\rho^2+2\rho+1-\frac{2\rho}{(1+\rho)^2}\right]\\ &= 4\rho^2 - \frac{2(\rho-1)(2\rho^5+12\rho^4+15\rho^3+13\rho^2+5\rho+1)}{2(1+\rho)^2(1+\rho+\rho^2)^2}\\ &\le 4\rho^2,\end{aligned} \tag{30}$$

$$\left|\frac{dw(i)}{v(i)\,dz}\right|^2 \ge \frac{4}{(K+1)^2}\cdot\left|\frac{w_z(i)}{v(i)}\right|^2 \ge \frac{4}{(2\rho+1)^2}\cdot\frac{(1+\rho)^2}{4\rho^2} \ge \frac{1}{4\rho^2}, \tag{31}$$

and that completes the proof.

Finally I want to express my heartfelt thanks to Professor Clifford Earle for his careful reading and many suggestions which made this paper better than the original one.

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Department of Mathematics, State University of New York at Stony Brook, Stony Brook, New York 11794