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A Q-MANIFOLD LOCAL-COMPACTIFICATION OF A METRIC COMBINATORIAL ∞-MANIFOLD

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ABSTRACT. Let K be a combinatorial ∞-manifold, that is, a countable simplicial complex such that the star of each vertex is combinatorially equivalent to the countable-infinite full simplicial complex. Then the space $|K|_m$ with the metric topology is a manifold modeled on the space σ, where σ is the subspace of the Hilbert cube $Q = I^\omega$ which consists of all points having at most finitely many nonzero coordinates. In this paper, we give a local-compactification of $|K|_m$ which is a $[0,1)$-stable Q-manifold containing $|K|_m$ as an f.d. cap set.

0. Introduction. Let X be a metric space with a metric d. A closed subset A of X is called a *Z-set in X* if for each $\varepsilon > 0$ and each map $f\colon I^n \to X, n \in \mathbf{N}$, there is a map $g\colon I^n \to X \setminus A$ with $d(f,g) < \varepsilon$. An *f.d. cap set for X* is a subset M of X such that $M = \bigcup_{n=1}^\infty M_n$ where $M_1 \subset M_2 \subset \cdots$ is a tower of finite-dimensional compact Z-sets in X and for each $\varepsilon > 0$, each $m \in \mathbf{N}$, and each finite-dimensional compact subset A of X, there is an $n \in \mathbf{N}$ and an embedding $h\colon A \to M_n$ such that $h|A \cap M_m = \mathrm{id}$ and $d(h, \mathrm{id}) < \varepsilon$ [**An** and **Ch**$_1$]. Let σ denote the subspace of the Hilbert cube $Q = I^\omega$ which consists of all points having at most finitely many nonzero coordinates. R. D. Anderson [**An**] showed that σ is an f.d. cap set for Q and that for any f.d. cap set M for Q there is a homeomorphism of Q onto itself taking M onto σ. A *Hilbert cube manifold* (or *Q-manifold*) is a separable manifold modeled on the Hilbert cube Q. A separable manifold modeled on the space σ is called a *σ-manifold.* T. A. Chapman [**Ch**$_1$] proved that any f.d. cap set for a Q-manifold is a σ-manifold and any σ-manifold can be embedded as an f.d. cap set for a Q-manifold and that for any f.d. cap sets M and N for a Q-manifold X, there is a homeomorphism of X onto itself taking M onto N.

Two simplicial complexes are said to be *combinatorially equivalent* provided they admit simplicially isomorphic subdivisions. A *combinatorial ∞-manifold* is a countable simplicial complex such that the star of each vertex is combinatorially equivalent to the countable-infinite full simplicial complex, namely a ∞-simplex [**Sa**$_1$]. For a simplicial complex K, $|K|$ denotes the underlying set and $|K|_m$ denotes the space $|K|$ with the topology induced by the metric

$$d_1(x,y) = \sum_{v \in K^0} |x(v) - y(v)|,$$

where $(x(v))_{v\in K^0}$ and $(y(v))_{v\in K^0}$ are the barycentric coordinates of x and y respectively. The author [**Sa**$_2$, **Sa**$_3$] showed that any σ-manifold is homeomorphic to

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$|K|_m$ for some combinatorial ∞-manifold K and that a countable simplicial complex K is a combinatorial ∞-manifold if and only if $|K|_m$ is a σ-manifold. For a combinatorial ∞-manifold K, we give here a local-compactification of $|K|_m$ which is a Q-manifold containing $|K|_m$ as an f.d. cap set.

Through the paper, *K always denotes a countable-infinite simplicial complex with vertices v_i, $i \in \mathbf{N}$.* For each $x \in |K|$, we write $x_i = x(v_i)$, $i \in \mathbf{N}$. Identifying $x \in |K|$ with $(x_i)_{i\in\mathbf{N}} \in Q$, we consider $|K| \subset Q$. The topology of $|K|_m$ is clearly the subspace topology on $|K|$. Thus $|K|_m$ can be considered as a subspace of Q. Then $|K|_m$ has a local-compactification

$$\overline{|K|}^{Q^*} = \mathrm{cl}_{Q\setminus\{0\}}|K| = (\mathrm{cl}_Q|K|) \setminus \{0\},$$

where 0 denotes the point $(0, 0, \dots) \in Q$. It is not difficult to see that $\overline{|K|}^{Q^*} = |K|$ *if and only if K is locally finite.* In §1, we show that $\overline{|K|}^{Q^*}$ *an ANR local-compactification of $|K|_m$ if K has no principal (maximal) simplex.* Our main result is the following which is proved in §2.

MAIN THEOREM. *If K is a combinatorial ∞-manifold, then $\overline{|K|}^{Q^*}$ is a $[0,1)$-stable Q-manifold and $|K|_m$ is an f.d. cap set for $\overline{|K|}^{Q^*}$.*

Here we say that a Q-manifold X is $[0,1)$-*stable* if $X \times [0,1)$ is homeomorphic to X. T. A. Chapman [**Ch**$_2$] showed that $[0,1)$-stable Q-manifolds are topologically classified by homotopy type. From this result and the topological uniqueness of f.d. cap sets for a Q-manifold, it follows that *if two combinatorial ∞-manifolds K and L have the same homotopy type, then the pairs $(\overline{|K|}^{Q^*}, |K|_m)$ and $(\overline{|L|}^{Q^*}, |L|_m)$ are homeomorphic.*

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1. The compactification $\mathrm{cl}_Q|K|$. We use the metric d_Q for Q defined by

$$d_Q(x,y) = \sup_{i\in\mathbf{N}} \min\{|x_i - y_i|, i^{-1}\}.$$

1.1. LEMMA. *If K has no principal simplex, then*

$$\mathrm{cl}_Q|K| = \left\{ x \in Q \mid \ (1)\ \sum_{i=1}^{\infty} x_i \le 1 \text{ and } \right.$$
$$\left. (2)\ v_{i_1}, \dots, v_{i_n} \text{ span a simplex of } K \text{ if } x_{i_1}, \dots, x_{i_n} \ne 0 \right\}.$$

PROOF. It is easy to see that each $x \in \mathrm{cl}_Q|K|$ satisfies (1) and (2). (Cf. [**Sa**$_4$, Lemma 2.2]. This is valid without hypothesis.) Conversely take $x \in Q$ satisfying (1) and (2). For each $m \in \mathbf{N}$, let

$$\{v_{i_1}, \dots, v_{i_n}\} = \{v_i \mid x_i \ne 0,\ i \le m\}.$$

Then $v_{i_1}, \dots, v_{i_n}$ span a simplex of K by (2). Since K has no principal simplex, we have $v_{i_{n+1}} \in K^0 \setminus \{v_{i_1}, \dots, v_{i_n}\}$ such that $v_{i_1}, \dots, v_{i_{n+1}}$ span a simplex of K and $i_{n+1} > m$. Note $x_{i_1} + \dots + x_{i_n} \le 1$ by (1). We define $y \in |K|$ as follows:

$$y_i = \begin{cases} x_{i_j} & \text{if } i = i_j,\ j = 1, \dots, n, \\ 1 - (x_{i_1} + \dots + x_{i_n}) & \text{if } i = i_{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $x_i = y_i$ for each $i = 1, \ldots, m$, we have $d_Q(x, y) < m^{-1}$. Therefore $x \in \mathrm{cl}_Q|K|$. □

1.2. THEOREM. *If K has no principal simplex, then $\mathrm{cl}_Q|K|$ is an AR; hence $\overline{|K|}^{Q^*}$ is an ANR.*

PROOF. Let $\mu\colon Q \times Q \to Q$ be a map defined by

$$\mu(x, y)_i = \min\{x_i, y_i\}, \qquad i \in \mathbf{N}.$$

Define a map $\lambda\colon Q \times Q \times I \to Q$ as follows:

$$\lambda(x, y, t) = \begin{cases} (1 - 2t)x + 2t\mu(x, y) & \text{if } 0 \le t \le \frac{1}{2}, \\ (2t - 1)y + (2 - 2t)\mu(x, y) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then $\lambda(x, y, 0) = x$, $\lambda(x, y, 1) = y$ and $\lambda(x, x, t) = x$ for any $x, y \in Q$ and $t \in I$, that is, λ is an equi-connecting map for Q. Using Lemma 1.1, it is easily seen that

$$\lambda(\mathrm{cl}_Q|K| \times \mathrm{cl}_Q|K| \times I) \subset \mathrm{cl}_Q|K|.$$

Let $x, y, z \in \mathrm{cl}_Q|K|$ such that $d_Q(x, z), d_Q(y, z) < n^{-1}$, and let $t \in I$. Then for each $i = 1, \ldots, n$,

$$|x_i - z_i|, |y_i - z_i| < n^{-1} \quad (\le i^{-1}),$$

which implies

$$|\mu(x, y)_i - z_i| = |\min\{x_i, y_i\} - z_i| < n^{-1}.$$

From the definition of λ, it follows that

$$|\lambda(x, y, t)_i - z_i| < n^{-1} \quad \text{for each } i = 1, \ldots, n.$$

Therefore we have

$$d_Q(\lambda(x, y, t), z) = \sup_{i \in \mathbf{N}} \min\{|\lambda(x, y, t)_i - z_i|, i^{-1}\} < n^{-1}.$$

This means the n^{-1}-neighborhood of z in $\mathrm{cl}_Q|K|$ is λ-convex. By the result of [**Du**], $\mathrm{cl}_Q|K|$ is an AR. □

1.3. REMARKS. The hypothesis of Theorem 1.2 is essential. In general $\overline{|K|}^{Q^*}$ is not even locally connected. In fact, if K is the cone over the positive integers then $\mathrm{cl}_Q|K|$ is homeomorphic to the cone over $\{0\} \cup \{1/n \mid n \in \mathbf{N}\}$. However, it is not a necessary condition. Let K be the cone over the complex of the half-open interval $[0, \infty)$. Then $\mathrm{cl}_Q|K|$ is homeomorphic to a 2-simplex, hence is an AR.

Under the hypothesis of Theorem 1.2, it can also be shown that *the inclusion $|K|_m \subset \overline{|K|}^{Q^*}$ is a weak homotopy equivalence, therefore a homotopy equivalence* (cf. [$\mathbf{Sa}_4$, Theorem 2.12]).

2. Proof of the Main Theorem and some remarks. First we prove the following

2.1. PROPOSITION. *Let X be an ANR local-compactification of a σ-manifold M and let d be a metric for X. If for each $\varepsilon > 0$ and each compact subset A of X there is a map $h\colon A \to M$ with $d(h, \mathrm{id}) < \varepsilon$, then X is a Q-manifold and M is an f.d. cap set for X.*

PROOF. H. Toruńczyk [**To**] characterized Q-manifolds as locally compact ANR's X with the disjoint approximation property (i.e., any two maps from Q to X can be

arbitrarily closely approximated by maps whose images are disjoint). The disjoint approximation property for the σ-manifold M (cf. [**Mo**]) and the mapping hypothesis imply that X also has the disjoint approximation property and is therefore a Q-manifold.

Next we will show that M is an f.d. cap set for X. By [$\mathbf{Sa}_2$, Lemma 2], M has a strongly universal tower $\{M_i\}_{i\in\mathbf{N}}$ for finite-dimensional compacta such that $M = \bigcup_{i\in\mathbf{N}} M_i$ and each M_i is a finite-dimensional compact (strong) Z-set in M (cf. [**CDM**]). Then the mapping hypothesis implies that each M_i is a Z-set in X. To verify the absorption property of the tower $\{M_i\}_{i\in\mathbf{N}}$, let A be finite-dimensional compact subset of X and let $m \in \mathbf{N}$ and $\varepsilon > 0$. Since M is an ANR, there exists $\delta > 0$ such that any map $f\colon A \cap M_m \to M$ with $d(f, \mathrm{id}) < \delta$ is $\varepsilon/3$-homotopic to the inclusion $A \cap M_m \subset M$. From the hypothesis, there is a map $h\colon A \to M$ with $d(h, \mathrm{id}) < \delta$. Then $h|A \cap M_m$ is $\varepsilon/3$-homotopic to the inclusion $A \cap M_m \subset M$. By the Homotopy Extension Theorem [**Hu**, Chapter IV, Theorem 2.2 and its proof], h is $\varepsilon/3$-homotopic to a map $j\colon A \to M$ with $j|A \cap M_m = \mathrm{id}$. From the strong universality of the tower $\{M_i\}_{i\in\mathbf{N}}$, we have an embedding $g\colon A \to M_n$ for some $n \geq m$ such that $g|A \cap M_m = j|A \cap M_m = \mathrm{id}$ and $d(g, j) < \varepsilon/3$. Then $d(g, \mathrm{id}) \leq d(g, j) + d(j, h) + d(h, \mathrm{id}) < \varepsilon$. Hence M is an f.d. cap set for X. $\square$

PROOF OF THE MAIN THEOREM. To verify the hypothesis of Proposition 2.1, let A be a compact subset of $\overline{|K|}^{Q^*}$ and $\varepsilon > 0$. Since A is closed in Q and $0 \notin A$, we have $n \in \mathbf{N}$ and $0 < s < 1$ such that $n^{-1} < \varepsilon/3$ and each $x \in A$ has a coordinate $x_i \geq s$ for some $i = 1, \ldots, n$. Let L be the finite full subcomplex for K with $\{v_1, \ldots, v_n\} = L^0$, and set

$$|L|^* = \{tx \mid x \in |L|, t \in [s, 1]\} \subset \overline{|K|}^{Q^*}.$$

Define a map $f\colon A \to |L|^*$ by

$$f(x)_i = \begin{cases} x_i & \text{for } i \leq n, \\ 0 & \text{for } i > n. \end{cases}$$

Then $d_Q(f, \mathrm{id}) < n^{-1} < \varepsilon/3$. Since $|L|$ is a compact subset of the σ-manifold $|K|_m$, $|L|$ is a Z-set in $|K|_m$. Hence there is a map $g\colon |K|_m \to |K|_m \setminus |L|$ with $d_Q(g, \mathrm{id}) < \varepsilon/3$. Since we have a homeomorphism $h\colon [s, 1] \times |L| \to |L|^*$ defined by $h(t, x) = tx$, we can define a map $g^*\colon |L|^* \to \overline{|K|}^{Q^*}$ by $g^*(tx) = tg(x)$. Then $d_Q(g^*, \mathrm{id}) \leq d_Q(g, \mathrm{id}) < \varepsilon/3$. Observe that

$$g^*(|L|^*) \cap \{tx \mid x \in |L|, t \in [0, 1]\} = \varnothing.$$

This implies that each $x \in g^*(|L|^*)$ has a nonzero coordinate $x_i \neq 0$ for some $i > n$. From compactness of $g^*(|L|^*)$, there exists $m > n$ such that each $x \in g^*(|L|^*)$ has a nonzero coordinate $x_i \neq 0$ for some $i = n+1, \ldots, m$. Define a map $k\colon g^*(|L|^*) \to |K|_m$ as follows:

$$k(x)_i = \begin{cases} x_i & \text{for } i \leq n, \\ \left(1 - \sum_{j=1}^n x_j\right) \cdot x_i / \sum_{j=n+1}^m x_j & \text{for } n < i \leq m, \\ 0 & \text{for } i > m. \end{cases}$$

Then $d_Q(k, \mathrm{id}) < n^{-1} < \varepsilon/3$. Thus we have a map $kg^*f\colon A \to |K|_m$. And then

$$d_Q(kg^*f, \mathrm{id}) \leq d_Q(k, \mathrm{id}) + d_Q(g^*, \mathrm{id}) + d_Q(f, \mathrm{id}) < \varepsilon.$$

By Proposition 2.1, $\overline{|K|}^{Q^*}$ is a Q-manifold and $|K|_m$ is an f.d. cap set for $\overline{|K|}^{Q^*}$.

To verify the $[0,1)$-stability of $\overline{|K|}^{Q^*}$, we use Wong's criterion [**Wo**]. Thus we show that $\overline{|K|}^{Q^*}$ is properly contractible to infinity; that is, for each compact subset A of $\overline{|K|}^{Q^*}$ there is a proper homotopy $h\colon \overline{|K|}^{Q^*} \times I \to \overline{|K|}^{Q^*}$ with $h_0 = \mathrm{id}$ and $h_1(\overline{|K|}^{Q^*}) \subset \overline{|K|}^{Q^*} \setminus A$. For any compact subset A of $\overline{|K|}^{Q^*}$, we have $n \in \mathbf{N}$ and $0 < s < 1$ such that

$$\{x \in Q \mid x_i \leq s \text{ for each } i = 1, \ldots, n\} \cap A = \varnothing.$$

Define a homotopy $h\colon Q \times I \to Q$ by

$$h(x,t) = ((1-t) + ts) \cdot x.$$

Then $h_0 = \mathrm{id}$ and $h_1(Q) \cap A = \varnothing$. By Lemma 1.1, $h(\overline{|K|}^{Q^*} \times I) \subset \overline{|K|}^{Q^*}$. Thus we have a proper homotopy $h\colon \overline{|K|}^{Q^*} \times I \to \overline{|K|}^{Q^*}$ with the desired property. Therefore $\overline{|K|}^{Q^*}$ is $[0,1)$-stable. □

It is natural to ask the following

2.2. QUESTION. *If $\overline{|K|}^{Q^*}$ is a Q-manifold, then is K a combinatorial ∞-manifold?*

From the above proof, it follows that *$\overline{|K|}^{Q^*}$ is $[0,1)$-stable if $\overline{|K|}^{Q^*}$ is a Q-manifold.*

For the countable-infinite full simplicial complex Δ^∞, $\mathrm{cl}_Q|\Delta^\infty|$ is homeomorphic to Q by Keller's Theorem [**Ke**]. For each $n \in \mathbf{N}$, the identity map of $\mathrm{cl}_Q|\Delta^\infty|$ is n^{-1}-close to a map $f\colon \mathrm{cl}_Q|\Delta^\infty| \to |\Delta^\infty|_m$ defined by

$$f(x)_i = \begin{cases} x_i & \text{for } i \leq n, \\ 1 - \sum_{i=1}^n x_i & \text{for } i = n+1, \\ 0 & \text{for } i > n+1. \end{cases}$$

Then as in the proof of Proposition 2.1, $|\Delta^\infty|_m$ is an f.d. cap set for $\mathrm{cl}_Q|\Delta^\infty|$. Thus the pair $(\mathrm{cl}_Q|\Delta^\infty|, |\Delta^\infty|_m)$ is homeomorphic to the pair (Q, σ). More generally, the following follows from the Main Theorem.

2.3. COROLLARY. *For any contractible combinatorial ∞-manifold K, the pair $(\mathrm{cl}_Q|K|, |K|_m)$ is homeomorphic to the pair (Q, σ).*

PROOF. By the Main Theorem, $\overline{|K|}^{Q^*}$ is homeomorphic to $Q \times [0,1)$ which is homeomorphic to $Q \setminus \{0\}$. Since $\mathrm{cl}_Q|K|$ and Q are the one-point compactifications of $\overline{|K|}^{Q^*}$ and $Q \setminus \{0\}$ respectively, they are homeomorphic. Then $\{0\}$ is a Z-set in $\mathrm{cl}_Q|K|$. Since $|K|_m$ is an f.d. cap set for $\overline{|K|}^{Q^*} = (\mathrm{cl}_Q|K|) \setminus \{0\}$, it is also an f.d. cap set for $\mathrm{cl}_Q|K|$. Thus $(\mathrm{cl}_Q|K|, |K|_m)$ is homeomorphic to (Q, σ). □

However, for a noncontractible combinatorial ∞-manifold K, $\mathrm{cl}_Q|K|$ is not a Q-manifold because $0 \in \mathrm{cl}_Q|K|$ and $\{0\}$ is not a Z-set in $\mathrm{cl}_Q|K|$ (cf. [**Sa**$_4$]).

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