A Q-MANIFOLD LOCAL-COMPACTIFICATION OF A METRIC COMBINATORIAL ∞-MANIFOLD
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ABSTRACT. Let K be a combinatorial ∞-manifold, that is, a countable simplicial complex such that the star of each vertex is combinatorially equivalent to the countable-infinite full simplicial complex. Then the space $|K|_m$ with the metric topology is a manifold modeled on the space $\sigma$, where $\sigma$ is the subspace of the Hilbert cube $Q = I^\omega$ which consists of all points having at most finitely many nonzero coordinates. In this paper, we give a local-compactification of $|K|_m$ which is a $[0, 1)$-stable Q-manifold containing $|K|_m$ as an f.d. cap set.

0. Introduction. Let X be a metric space with a metric d. A closed subset A of X is called a Z-set in X if for each $\varepsilon > 0$ and each map $f : I^n \to X, n \in \mathbb{N}$, there is a map $g : I^n \to X \setminus A$ with $d(f, g) < \varepsilon$. An f.d. cap set for X is a subset M of X such that $M = \bigcup_{n=1}^{\infty} M_n$ where $M_1 \subset M_2 \subset \cdots$ is a tower of finite-dimensional compact Z-sets in X and for each $\varepsilon > 0$, each $m \in \mathbb{N}$, and each finite-dimensional compact subset A of X, there is an $n \in \mathbb{N}$ and an embedding $h : A \to M_n$ such that $h|A \cap M_m = \text{id}$ and $d(h, \text{id}) < \varepsilon$ [An and Ch1]. Let $\sigma$ denote the subspace of the Hilbert cube $Q = I^\omega$ which consists of all points having at most finitely many nonzero coordinates. R. D. Anderson [An] showed that $\sigma$ is an f.d. cap set for Q and that for any f.d. cap set M for Q there is a homeomorphism of Q onto itself taking M onto $\sigma$. A Hilbert cube manifold (or Q-manifold) is a separable manifold modeled on the Hilbert cube Q. A separable manifold modeled on the space $\sigma$ is called a $\sigma$-manifold. T. A. Chapman [Ch1] proved that any f.d. cap set for a Q-manifold is a $\sigma$-manifold and any $\sigma$-manifold can be embedded as an f.d. cap set for a Q-manifold and that for any f.d. cap sets M and N for a Q-manifold X, there is a homeomorphism of X onto itself taking M onto N.

Two simplicial complexes are said to be combinatorially equivalent provided they admit simplicially isomorphic subdivisions. A combinatorial ∞-manifold is a countable simplicial complex such that the star of each vertex is combinatorially equivalent to the countable-infinite full simplicial complex, namely a $\infty$-simplex [Sa1]. For a simplicial complex K, $|K|$ denotes the underlying set and $|K|_m$ denotes the space $|K|$ with the topology induced by the metric

$$d_1(x, y) = \sum_{v \in K^\circ} |x(v) - y(v)|,$$

where $(x(v))_{v \in K^\circ}$ and $(y(v))_{v \in K^\circ}$ are the barycentric coordinates of x and y respectively. The author [Sa2, Sa3] showed that any $\sigma$-manifold is homeomorphic to
\(|K|_m\) for some combinatorial \(\infty\)-manifold \(K\) and that a countable simplicial complex \(K\) is a combinatorial \(\infty\)-manifold if and only if \(|K|_m\) is a \(\sigma\)-manifold. For a combinatorial \(\infty\)-manifold \(K\), we give here a local-compactification of \(|K|_m\) which is a \(Q\)-manifold containing \(|K|_m\) as an f.d. cap set.

Through the paper, \(K\) always denotes a countable-infinite simplicial complex with vertices \(v_i, i \in \mathbb{N}\). For each \(x \in \{|K|\}\), we write \(x_i = x(v_i), i \in \mathbb{N}\). Identifying \(x \in \{|K|\}\) with \((x_i)_{i \in \mathbb{N}} \in \mathbb{Q}\), we consider \(|K| \subset \mathbb{Q}\). The topology of \(|K|_m\) is clearly the subspace topology on \(|K|\). Thus \(|K|_m\) can be considered as a subspace of \(\mathbb{Q}\). Then \(|K|_m\) has a local-compactification

\[
|K|^{Q^*} = cl_Q \{|K| \} = (cl_Q |K|) \setminus \{0\},
\]

where 0 denotes the point \((0,0,\ldots) \in \mathbb{Q}\). It is not difficult to see that \(|K|^{Q^*} = |K|\) if and only if \(K\) is locally finite. In §1, we show that \(|K|^{Q^*}\) an ANR local-compactification of \(|K|_m\) if \(K\) has no principal (maximal) simplex. Our main result is the following which is proved in §2.

**Main Theorem.** If \(K\) is a combinatorial \(\infty\)-manifold, then \(|K|^{Q^*}\) is a \([0,1)\)-stable \(Q\)-manifold and \(|K|_m\) is an f.d. cap set for \(|K|^{Q^*}\).

Here we say that a \(Q\)-manifold \(X\) is \([0,1)\)-stable if \(X \times [0,1)\) is homeomorphic to \(X\). T. A. Chapman [Ch2] showed that \([0,1)\)-stable \(Q\)-manifolds are topologically classified by homotopy type. From this result and the topological uniqueness of f.d. cap sets for a \(Q\)-manifold, it follows that if two combinatorial \(\infty\)-manifolds \(K\) and \(L\) have the same homotopy type, then the pairs \((|K|^{Q^*}, |K|_m)\) and \((|L|^{Q^*}, |L|_m)\) are homeomorphic.

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1. **The compactification** \(cl_Q |K|\). We use the metric \(d_Q\) for \(\mathbb{Q}\) defined by

\[
d_Q(x,y) = \sup_{i \in \mathbb{N}} \min\{|x_i - y_i|, i^{-1}\}.
\]

1.1. **Lemma.** If \(K\) has no principal simplex, then

\[
cl_Q |K| = \left\{ x \in \mathbb{Q} \mid (1) \sum_{i=1}^{\infty} x_i \leq 1 \text{ and } (2) \text{ } v_{i_1}, \ldots, v_{i_n} \text{ span a simplex of } K \text{ if } x_{i_1}, \ldots, x_{i_n} \neq 0 \right\}.
\]

**Proof.** It is easy to see that each \(x \in cl_Q |K|\) satisfies (1) and (2). (Cf. [Sa4, Lemma 2.2]. This is valid without hypothesis.) Conversely take \(x \in \mathbb{Q}\) satisfying (1) and (2). For each \(m \in \mathbb{N}\), let

\[
\{v_{i_1}, \ldots, v_{i_m}\} = \{v_i \mid x_i \neq 0, i \leq m\}.
\]

Then \(v_{i_1}, \ldots, v_{i_m}\) span a simplex of \(K\) by (2). Since \(K\) has no principal simplex, we have \(v_{i_{n+1}} \in \{v_{i_1}, \ldots, v_{i_n}\}\) such that \(v_{i_1}, \ldots, v_{i_{n+1}}\) span a simplex of \(K\) and \(i_{n+1} > m\). Note \(x_{i_1} + \cdots + x_{i_n} \leq 1\) by (1). We define \(y \in |K|\) as follows:

\[
y_i = \begin{cases} x_i & \text{if } i = i_j, j = 1, \ldots, n, \\ 1 - (x_{i_1} + \cdots + x_{i_n}) & \text{if } i = i_{n+1}, \\ 0 & \text{otherwise.} \end{cases}
\]
Since $x_i = y_i$ for each $i = 1, \ldots, m$, we have $d_Q(x, y) < m^{-1}$. Therefore $x \in \text{cl}_Q[K]$. □

1.2. Theorem. If $K$ has no principal simplex, then $\text{cl}_Q[K]$ is an AR; hence $\text{cl}_Q[K]^0$ is an ANR.

Proof. Let $\mu : Q \times Q \to Q$ be a map defined by

$$\mu(x, y)_i = \min\{x_i, y_i\}, \quad i \in \mathbb{N}.$$ 

Define a map $\lambda : Q \times Q \times I \to Q$ as follows:

$$\lambda(x, y, t) = \begin{cases} (1 - 2t)x + 2t\mu(x, y) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (2t - 1)y + (2 - 2t)\mu(x, y) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then $\lambda(x, y, 0) = x$, $\lambda(x, y, 1) = y$ and $\lambda(x, x, t) = x$ for any $x, y \in Q$ and $t \in I$, that is, $\lambda$ is an equi-connecting map for $Q$. Using Lemma 1.1, it is easily seen that $\lambda(\text{cl}_Q[K] \times \text{cl}_Q[K] \times I) \subset \text{cl}_Q[K]$.

Let $x, y, z \in \text{cl}_Q[K]$ such that $d_Q(x, z), d_Q(y, z) < n^{-1}$, and let $t \in I$. Then for each $i = 1, \ldots, n$,

$$|x_i - z_i|, |y_i - z_i| < n^{-1} \quad (\leq i^{-1}),$$

which implies

$$|\mu(x, y)_i - z_i| = |\min\{x_i, y_i\} - z_i| < n^{-1}.$$ 

From the definition of $\lambda$, it follows that

$$|\lambda(x, y, t)_i - z_i| < n^{-1} \quad \text{for each } i = 1, \ldots, n.$$ 

Therefore we have

$$d_Q(\lambda(x, y, t), z) = \sup_{i \in \mathbb{N}} \min\{|\lambda(x, y, t)_i - z_i|, i^{-1}\} < n^{-1}.$$ 

This means the $n^{-1}$-neighborhood of $z$ in $\text{cl}_Q[K]$ is $\lambda$-convex. By the result of [Du], $\text{cl}_Q[K]$ is an AR. □

1.3. Remarks. The hypothesis of Theorem 1.2 is essential. In general $\text{cl}_Q[K]^0$ is not even locally connected. In fact, if $K$ is the cone over the positive integers then $\text{cl}_Q[K]$ is homeomorphic to the cone over $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$. However, it is not a necessary condition. Let $K$ be the cone over the complex of the half-open interval $[0, \infty)$. Then $\text{cl}_Q[K]$ is homeomorphic to a 2-simplex, hence is an AR.

Under the hypothesis of Theorem 1.2, it can also be shown that the inclusion $\text{cl}_Q[K] \subset [K]^Q$ is a weak homotopy equivalence, therefore a homotopy equivalence (cf. [Sa4, Theorem 2.12]).

2. Proof of the Main Theorem and some remarks. First we prove the following

2.1. Proposition. Let $X$ be an ANR local-compactification of a $\sigma$-manifold $M$ and let $d$ be a metric for $X$. If for each $\varepsilon > 0$ and each compact subset $A$ of $X$ there is a map $h : A \to M$ with $d(h, \text{id}) < \varepsilon$, then $X$ is a $Q$-manifold and $M$ is an $f.d.\ cap$ set for $X$.

Proof. H. Toruńczyk [To] characterized $Q$-manifolds as locally compact ANR’s $X$ with the disjoint approximation property (i.e., any two maps from $Q$ to $X$ can be
arbitrarily closely approximated by maps whose images are disjoint). The disjoint approximation property for the σ-manifold $M$ (cf. [Mo]) and the mapping hypothesis imply that $X$ also has the disjoint approximation property and is therefore a $Q$-manifold.

Next we will show that $M$ is an f.d. cap set for $X$. By [Sa2, Lemma 2], $M$ has a strongly universal tower $\{M_t\}_{t \in \mathbb{N}}$ for finite-dimensional compacta such that $M = \bigcup_{t \in \mathbb{N}} M_t$ and each $M_t$ is a finite-dimensional compact (strong) $Z$-set in $M$ (cf. [CDM]). Then the mapping hypothesis implies that each $M_t$ is a $Z$-set in $X$. To verify the absorption property of the tower $\{M_t\}_{t \in \mathbb{N}}$, let $A$ be finite-dimensional compact subset of $X$ and let $m \in \mathbb{N}$ and $\varepsilon > 0$. Since $M$ is an ANR, there exists $\delta > 0$ such that any map $f: \Delta^n \cap M_m \rightarrow M$ with $d(f, id) < \delta$ is $\varepsilon/3$-homotopic to the inclusion $\Delta^n \cap M_m \subset M$. From the hypothesis, there is a map $h: \Delta^n \rightarrow M$ with $d(h, id) < \delta$. Then $h|\Delta^n \cap M_m$ is $\varepsilon/3$-homotopic to the inclusion $\Delta^n \cap M_m \subset M$. By the Homotopy Extension Theorem [Hu, Chapter IV, Theorem 2.2 and its proof], $h$ is $\varepsilon/3$-homotopic to a map $j: \Delta^n \rightarrow M$ with $j|\Delta^n \cap M_m = id$. From the strong universality of the tower $\{M_t\}_{t \in \mathbb{N}}$, we have an embedding $g: \Delta^n \rightarrow M_n$ for some $n \geq m$ such that $g|\Delta^n \cap M_m = j|\Delta^n \cap M_m = id$ and $d(g, j) < \varepsilon/3$. Then $d(g, id) \leq d(g, j) + d(j, h) + d(h, id) < \varepsilon$. Hence $M$ is an f.d. cap set for $X$. \qed

**Proof of the Main Theorem.** To verify the hypothesis of Proposition 2.1, let $A$ be a compact subset of $|\overline{K}|^Q^*$ and $\varepsilon > 0$. Since $A$ is closed in $Q$ and $0 \notin A$, we have $n \in \mathbb{N}$ and $0 < s < 1$ such that $n^{-1} < \varepsilon/3$ and each $x \in A$ has a coordinate $x_i \geq s$ for some $i = 1, \ldots, n$. Let $L$ be the finite full subcomplex for $K$ with $\{v_1, \ldots, v_n\} = L^0$, and set

$$|L|^* = \{tx \mid x \in |L|, t \in [s, 1]\} \subset |\overline{K}|^Q^*.$$  

Define a map $f: \A \rightarrow |L|^*$ by

$$f(x)_i = \begin{cases} x_i & \text{for } i \leq n, \\ 0 & \text{for } i > n. \end{cases}$$

Then $d_Q(f, id) < n^{-1} < \varepsilon/3$. Since $|L|$ is a compact subset of the σ-manifold $|K|_m$, $|L|$ is a $Z$-set in $|K|_m$. Hence there is a map $g: |K|_m \rightarrow |K|_m \setminus |L|$ with $d_Q(g, id) < \varepsilon/3$. Since we have a homeomorphism $h: [s, 1] \times |L| \rightarrow |L|^*$ defined by $h(t, x) = tx$, we can define a map $g^*: |L|^* \rightarrow |\overline{K}|^Q^*$ by $g^*(tx) = tg(x)$. Then $d_Q(g^*, id) < \varepsilon/3$. Observe that

$$g^*(|L|^*) \cap \{tx \mid x \in |L|, t \in [0, 1]\} = \emptyset.$$  

This implies that each $x \in g^*(|L|^*)$ has a nonzero coordinate $x_i \neq 0$ for some $i > n$. From compactness of $g^*(|L|^*)$, there exists $m > n$ such that each $x \in g^*(|L|^*)$ has a nonzero coordinate $x_i \neq 0$ for some $i = n+1, \ldots, m$. Define a map $k: g^*(|L|^*) \rightarrow |K|_m$ as follows:

$$k(x)_i = \begin{cases} x_i & \text{for } i \leq n, \\ (1 - \sum_{j=1}^n x_j) \cdot x_i / \sum_{j=n+1}^m x_j & \text{for } n < i \leq m, \\ 0 & \text{for } i > m. \end{cases}$$

Then $d_Q(k, id) < n^{-1} < \varepsilon/3$. Thus we have a map $kg^* f: \A \rightarrow |K|_m$. And then

$$d_Q(kg^* f, id) \leq d_Q(k, id) + d_Q(g^*, id) + d_Q(f, id) < \varepsilon.$$
By Proposition 2.1, $\overline{K}^Q$ is a $Q$-manifold and $|K|^m$ is an f.d. cap set for $\overline{K}^Q$.

To verify the $[0,1)$-stability of $\overline{K}^Q$, we use Wong's criterion [Wo]. Thus we show that $\overline{K}^Q$ is properly contractible to infinity; that is, for each compact subset $A$ of $\overline{K}^Q$ there is a proper homotopy $h: \overline{K}^Q \times I \to \overline{K}^Q$ with $h_0 = \text{id}$ and $h_1(\overline{K}^Q) \subset \overline{K}^Q \setminus A$. For any compact subset $A$ of $\overline{K}^Q$, we have $n \in \mathbb{N}$ and $0 < s < 1$ such that

$$\{x \in Q | x_i \leq s \text{ for each } i = 1, \ldots, n\} \cap A = \emptyset.$$ 

Define a homotopy $h: Q \times I \to Q$ by

$$h(x,t) = ((1-t) + ts) \cdot x.$$ 

Then $h_0 = \text{id}$ and $h_1(Q) \cap A = \emptyset$. By Lemma 1.1, $h(\overline{K}^Q \times I) \subset \overline{K}^Q$. Thus we have a proper homotopy $h: \overline{K}^Q \times I \to \overline{K}^Q$ with the desired property. Therefore $\overline{K}^Q$ is $[0,1)$-stable. □

It is natural to ask the following

2.2. QUESTION. If $\overline{K}^Q$ is a $Q$-manifold, then is $K$ a combinatorial $\infty$-manifold?

From the above proof, it follows that $\overline{K}^Q$ is $[0,1)$-stable if $\overline{K}^Q$ is a $Q$-manifold.

For the countable-infinite full simplicial complex $\Delta^\infty$, $\text{cl}_Q|\Delta^\infty|$ is homeomorphic to $Q$ by Keller's Theorem [Ke]. For each $n \in \mathbb{N}$, the identity map of $\text{cl}_Q|\Delta^\infty|$ is $n^{-1}$-close to a map $f: \text{cl}_Q|\Delta^\infty| \to |\Delta^\infty|^m$ defined by

$$f(x)_i = \begin{cases} x_i & \text{for } i \leq n, \\ 1 - \sum_{i=1}^{n} x_i & \text{for } i = n + 1, \\ 0 & \text{for } i > n + 1. \end{cases}$$

Then as in the proof of Proposition 2.1, $\text{cl}_Q|\Delta^\infty|^m$ is an f.d. cap set for $\text{cl}_Q|\Delta^\infty|$. Thus the pair $(\text{cl}_Q|\Delta^\infty|, |\Delta^\infty|^m)$ is homeomorphic to the pair $(Q, \sigma)$. More generally, the following follows from the Main Theorem.

2.3. COROLLARY. For any contractible combinatorial $\infty$-manifold $K$, the pair $(\text{cl}_Q|K|, |K|^m)$ is homeomorphic to the pair $(Q, \sigma)$.

PROOF. By the Main Theorem, $\overline{K}^{\infty}$ is homeomorphic to $Q \times [0,1)$ which is homeomorphic to $Q \setminus \{0\}$. Since $\text{cl}_Q|K|$ and $Q$ are the one-point compactifications of $\overline{K}^{\infty}$ and $Q \setminus \{0\}$ respectively, they are homeomorphic. Then $\{0\}$ is a $Z$-set in $\text{cl}_Q|K|$. Since $|K|^m$ is an f.d. cap set for $\overline{K}^{\infty} = (\text{cl}_Q|K|) \setminus \{0\}$, it is also an f.d. cap set for $\text{cl}_Q|K|$. Thus $(\text{cl}_Q|K|, |K|^m)$ is homeomorphic to $(Q, \sigma)$. □

However, for a noncontractible combinatorial $\infty$-manifold $K$, $\text{cl}_Q|K|$ is not a $Q$-manifold because $0 \in \text{cl}_Q|K|$ and $\{0\}$ is not a $Z$-set in $\text{cl}_Q|K|$ (cf. [Sa4]).

REFERENCES


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