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NORMS ON ENVELOPING ALGEBRAS

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ABSTRACT. Let $\mathfrak{g}$ be a complex Lie algebra, and let U be its enveloping algebra. U is normable if and only if $\mathfrak{g}$ is nilpotent.

Let A be a complex algebra. Then the following are clearly equivalent:

(1) A has an algebra norm.

(2) A can be embedded into a Banach algebra.

(3) A admits a faithful representation by bounded Banach space operators.

Several authors [**1**, **3**] studied the normability problem for A commutative. We consider here the case $A = U(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra. When $\mathfrak{g}$ is abelian, $U(\mathfrak{g})$ is a polynomial algebra, which is normable. On the other hand, when $\mathfrak{g}$ is semisimple, it is not difficult to show that $U(\mathfrak{g})$ has no submultiplicative norm (see [**4**, **5**]).

Our main result is the following

THEOREM. *$U(\mathfrak{g})$ has an algebra norm if and only if $\mathfrak{g}$ is nilpotent.*

PROOF. (1) *Only if.* Assume $\mathfrak{g}$ is not nilpotent. By Engel's theorem, there exists $x \in \mathfrak{g}$ such that $\operatorname{ad} x\colon \mathfrak{g} \to \mathfrak{g}$ is not nilpotent, so $\operatorname{ad} x$ has a nonzero eigenvalue λ with nonzero eigenvector y. This means that $[x, y] = \lambda y$. Now there cannot be a submultiplicative norm $\|\ \|$ on $U(\mathfrak{g})$, for $xy - yx = \lambda y$ implies $xy^k - y^k x = k\lambda y^k$ ($k \in \mathbf{N}$), whence $2\|x\|\,\|y^k\| \geq k|\lambda|\,\|y^k\|$ and $k \leq 2\|x\|/|\lambda|$ for all k, a contradiction.

(2) *If.* Suppose $\mathfrak{g}$ is nilpotent. Then [**2**, Theorem 2.5.5] $\mathfrak{g}$ can be embedded as a subalgebra of $\mathfrak{n}$, the Lie algebra of strictly triangular $n \times n$ matrices (for some n). Since $U(\mathfrak{g})$ embeds into $U(\mathfrak{n})$, it will be sufficient to show that $U(\mathfrak{n})$ has a faithful representation by bounded operators on a Banach space.

We consider the Lie algebras

$\mathfrak{s} = \mathfrak{gl}(n, C)$,

$\mathfrak{r} = \{x \in \mathfrak{s}\colon (x)_{ij} = 0 \text{ for } i > j\}$, and

$\mathfrak{n} = \{x \in \mathfrak{s}\colon (x)_{ij} = 0 \text{ for } i \geq j\} = [\mathfrak{r}, \mathfrak{r}]$.

Let $U(\mathfrak{s}) = \bigoplus_{d=0}^{\infty} U^d(\mathfrak{s})$ be the natural filtration. For every $d \in \mathbf{N}$, there exists a representation ϕ_d of $U(\mathfrak{s})$ acting on a finite dimensional Banach space E_d, such that $\ker \phi_d \cap U^d(\mathfrak{s}) = 0$ [**2**, Theorem 2.5.7]. By Lie's Theorem, ψ_d, the restriction of ϕ_d to $U(\mathfrak{r})$, can be triangularized, and ρ_d, the restriction to $U(\mathfrak{n})$, will be strictly triangular.

Note that $\ker \rho_d \cap U^d(\mathfrak{n}) = 0$. Our problem is to combine the ρ_d into a single, faithful representation. Let $y_i = e_{ii+1} \in \mathfrak{n}$ ($1 \leq i \leq n-1$). The y_i generate $\mathfrak{n}$ as a Lie algebra. Let $Y_i^d = \rho_d(y_i) \in \mathcal{L}(E_d)$. Since the Y_i^d are strictly triangular for a given d, we can arrange by a change of norm on E_d to have $\|Y_i^d\| \leq 1$ ($1 \leq i \leq n-1$).

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Do this for every d, let E be the l^1-direct sum of the E_d, and put $\rho = \rho_1 \oplus \rho_2 \oplus \cdots$ acting on E. Then $\|\rho(y_i)\|_{\mathcal{L}(E)} \leq 1$, and we get a map $U(\mathfrak{n}) \to \mathcal{L}(E)$ which is a faithful representation of $U(\mathfrak{n})$ by bounded operators on E.

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