PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 1, September 1987

ON THE CALKIN REPRESENTATIONS OF $B(\mathcal{H})$

H. BEHNCKE

Dedicated to H. Leptin on the occasion of his sixtieth birthday

ABSTRACT. Irreducible representations of $B(\mathcal{H})$, with $\mathcal{H}$ a not necessarily separable Hilbert space, are constructed and analyzed along the lines of a similar study of Reid for separable Hilbert spaces.

Here we construct and study certain representations of $B(\mathcal{H})$, the algebra of bounded operators on the Hilbert space $\mathcal{H}$. These have been studied previously by Calkin [**4**] for separable Hilbert spaces and in the general case by Barnes [**3**]. A more systematic study of these irreducible Calkin representations of $B(\mathcal{H})$, $\mathcal{H}$ separable, has been undertaken by Reid [**7**], but not much seems to be known in the nonseparable case beyond [**1**]. In this paper we extend the results of Reid to the nonseparable case.

Since such a study involves some cardinal arithmetic we shall assume Zorn's Lemma and the generalized continuum hypothesis throughout. For the notation and terminology regarding ordinals, cardinals, and filters we refer the reader to the book of Comfort and Negrepontis [**5**]. Our notation and terminology regarding C^*-algebras will be standard; i.e., that of [**6**]. If $\{\xi_s \mid s \in A\} \subset \mathcal{H}$, the range projection onto the linear span of the ξ_s, $s \in A$, will be denoted by $P = \langle \xi_s \mid s \in A\rangle$.

Let $\mathcal{H}$ be a Hilbert space of dimension α. Then the proper closed two-sided ideals of $B(\mathcal{H})$ are just the κ-compact operators I_κ, $\omega \le \kappa \le \alpha$. I_κ is generated by all projections of dimension strictly less than κ. Thus I_ω is the usual ideal of compact operators and for $B(\mathcal{H})$ we have the composition series

$$\{0\} \subset I_\omega \subset \cdots \subset I_\kappa \subset \cdots \subset I_\alpha \subset B(\mathcal{H}).$$

An irreducible representation π of $B(\mathcal{H})$ will thus have one of these ideals as the kernel.

The Calkin representations which we are going to study are all extensions of representations of the maximal abelian subalgebra $l^\infty(\alpha)$ of $B(\mathcal{H})$, $\dim \mathcal{H} = \alpha$. Remember that we identify the cardinal (ordinal) α with the corresponding set.

Let $\mathcal{H}^\alpha = l^\infty(\alpha, \mathcal{H})$ be the system of bounded $\mathcal{H}$-valued sequences indexed by α. If p is an ultrafilter on α, we can define an inner product on $\mathcal{H}^\alpha$ by

$$(1) \qquad \langle \xi, \eta\rangle_p = \lim_p \langle \xi_s, \eta_s\rangle \quad \text{for } \xi = (\xi_s),\ \eta = (\eta_s).$$

With $\mathcal{N}_p = \{\xi \in \mathcal{H}^\alpha \mid \langle \xi, \xi\rangle_p = \|\xi\|_p = 0\}$, $\mathcal{H}_p = H^\alpha/\mathcal{N}_p$ becomes a Hilbert space.

$B(\mathcal{H})$ operates on $\mathcal{H}^\alpha$ coordinatewise by

$$(2) \qquad (T\xi)_s = T\xi_s.$$

Received by the editors June 3, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L05.
Key words and phrases. Calkin representations, extension property.
The author acknowledges fruitful discussions and correspondence with J. Anderson.

This action leaves $\mathcal{N}_p$ invariant. Thus this allows us to define a *-representation of $B(\mathcal{H})$ on $\mathcal{H}_p$ by "coordinatewise action" from the left. This representation is far from irreducible, because the space $\tilde{\mathcal{H}}$ of constant sequences is invariant, and $\tilde{\mathcal{H}} = \mathcal{H}_p$ only if p is principal. In this case $\mathcal{H}_p \simeq \mathcal{H} = \tilde{\mathcal{H}}$ and the representation is equivalent to the identity representation.

In Calkin's construction the space $\mathcal{W}$ of vectors $\xi \in \mathcal{H}^\alpha$ converging to 0 weakly plays an important role. Since

$$\mathcal{W} = \{\xi \mid \left|\{s \mid |\langle \xi_s, \eta\rangle| > \varepsilon\right|\} < \omega \text{ for each } \eta \in \mathcal{H} \text{ and } \varepsilon > 0\}$$

we define for any cardinal κ with $\omega \leq \kappa < \alpha$

$$\mathcal{W}_\kappa = \{\xi \mid \left|\{s \mid \langle \xi_s, \eta\rangle \neq 0\}\right| \leq \kappa \text{ for each } \eta \in \mathcal{H}\}.$$

The spaces $\mathcal{W}$ and $\mathcal{W}_\kappa$ are clearly closed $B(\mathcal{H})$ invariant subspaces of $\mathcal{H}^\alpha$ and $\mathcal{W} \subset \mathcal{W}_\omega \subset \cdots \subset \mathcal{W}_\kappa$. Moreover we see that if $\xi \in \mathcal{W}_\kappa$ and P is a projection with $\dim P \leq \kappa$, then $\left|\{s \mid \|P\xi_s\| \neq 0\}\right| \leq \kappa$.

REMARK. In this notation $\mathcal{W}$ corresponds rather to the "cardinal" "finite" or "$< \omega$," and we could have defined $\mathcal{W}_{<\kappa}$ similarly. Since this however would lead to difficulties with irregular cardinals, we have not pursued this, though these cases could be treated by similar methods. In the proof however it is clear that irregular cardinals require a somewhat more careful argument though the underlying ideas are the same.

Identifying $\tilde{\mathcal{H}}$ with its image in $\mathcal{H}_p$, we see that $\tilde{\mathcal{H}}^\perp = \mathcal{W}_p/\mathcal{N}_p$, where $\mathcal{W}_p = \{\xi \in \mathcal{H}^\alpha \mid \text{w-lim}_p\, \xi_s = 0\}$.

LEMMA 1. *Let $\kappa \geq \omega$, let $\mathcal{A}$ be a κ-separable C^*-subalgebra of $B(\mathcal{H})$, and let $\xi = (\xi_s) \in \mathcal{W}_\kappa$. Then there is a decomposition of α into sets d_ρ, $0 \leq \rho < \alpha$, with $|d_\rho| \leq \kappa$ such that*

$$\mathcal{A}\xi_s \perp \mathcal{A}\xi_t \quad \textit{if } s \in d_\rho,\ t \in d_\sigma,\ \textit{and } \rho \neq \sigma.$$

PROOF. (a) We may assume $\mathcal{A}$ to be unital and $\|\xi\| = \|\xi_s\| = 1$ for all $s \in \alpha$. By assumption there is a set $\{a_i \mid i \in I\}$ dense in the unit ball of $\mathcal{A}$ with $|I| \leq \kappa$. We define inductively sets $J_n \subset \alpha$ such that the following properties hold:

(i) $|J_n| \leq \kappa$,

(ii) $J_n \subset J_{n+1}$,

(iii) $\langle \xi_s, a\xi_t\rangle = 0,\ a \in A,\ t \in J_n,\ s \notin J_{n+1}$.

$J_0 = \{s\}$, where $s \in \alpha$ is arbitrary. Now assume J_n has been constructed. Then let $J_{n+1} = \{s \mid \langle \xi_s, a_i\xi_t\rangle \neq 0,\ i \in I, t \in J_n\}$. Obviously $|J_{n+1}| \leq \kappa$, $J_n \subset J_{n+1}$, and by assumption $\langle \xi_s, a_i\xi_t\rangle = 0$ if $s \notin J_{n+1}$, $t \in J_n$. Let $J = \bigcup J_n$. Then $|J| \leq \kappa$ and $\langle a\xi_s, b\xi_t\rangle = \langle \xi_s, a^*b\xi_t\rangle = 0$ for $t \in J$ and $s \notin J$.

(b) With the aid of Zorn's Lemma this construction can now be completed.

For $\mathcal{W}$ a similar result is true; however, in this case the blocks have countable size.

Actually we are not so much interested in the spaces $\mathcal{W}$ and $\mathcal{W}_\kappa$ but rather in the spaces $\mathcal{K}_{0,p} = \mathcal{W}/\mathcal{W} \cap \mathcal{N}_p$ and $\mathcal{K}_{\kappa,p} = \mathcal{W}_\kappa/\mathcal{W}_\kappa \cap \mathcal{N}_p$. The same argument as in **[7]** now shows that the norm in $\mathcal{K}_{\kappa,p}$ is given by $\lim_p \|\xi_s\|$. Thus $\mathcal{K}_{0,p}$ and the spaces $\mathcal{K}_{\kappa,p}$ are actually closed $B(\mathcal{H})$-invariant spaces of $\mathcal{H}_p$. It shows moreover, that any representing vector $\xi = (\xi_s)$ can be chosen such that $\|\xi_s\| = \|\xi\|$ for all s. The representations of $B(\mathcal{H})$ on $\mathcal{K}_{\kappa,p}$ will be denoted by $\pi_{\kappa,p}$ with $\pi_{0,p} = \pi_p$.

Let $\kappa \leq \alpha$ be a cardinal and assume $\|p\| = \kappa' \leq \kappa$. Then $\tilde{\mathcal{H}} \subset \mathcal{W}_\kappa + \mathcal{N}_p$ and thus $\pi_{\kappa,p}$ is faithful. Thus we shall assume $\|p\| > \kappa$ in the remainder.

Now let $\|p\| = \kappa' > \kappa \geq \omega$ and let T be κ'-compact. Then $T\mathcal{W}_\kappa \subset \mathcal{N}_p$. In order to see this we may assume T to be a projection. Arguing as in the proof of Lemma 1 we see that $\left|\{s \in \alpha \mid T\xi_s \neq 0\}\right| = \max \operatorname{rank} T, \kappa$ for any $\xi = (\xi_s) \in \mathcal{W}_\kappa$. Thus $T\xi \in \mathcal{N}_p$. Now let $A \subset \alpha$ with $|A| = \kappa'$ and $A \in p$. Choose an orthonormal system $\{\xi_s \mid s \in A\}$ and set $\xi_s = 0$ for $s \notin A$. Then $0 \neq \xi = (\xi_s) \in \mathcal{W} \subset \mathcal{W}_\kappa$. Let $P = \langle \xi_s \mid s \in A \rangle$. Then $P\xi = \xi$. Thus the kernel of the representation of $B(\mathcal{H})$ on $\mathcal{K}_{\kappa,p}$ is $I_{\kappa'}$. Of course this extends immediately to representations on $\mathcal{K}_{0,p}$.

Thus we have

LEMMA 2. (a) *Let p be an ultrafilter with $\|p\| = \kappa'$ and let $\kappa \geq \kappa'$. Then* $\ker \pi_{\kappa,p} = (0)$.

(b) *Let $\|p\| = \kappa' > \kappa$. Then $I_{\kappa'} = \ker \pi_{\kappa,p}$.*

Of course we can also consider the representation of $B(\mathcal{H})$ on $\tilde{\mathcal{H}}^\perp$. For each $\xi \in \mathcal{W}_p$ and projection P of finite rank, $P\xi \in \mathcal{N}_p$ if p is nonprincipal. Thus this representation annihilates I_ω, but we cannot expect more. To see this let p be a countable incomplete nonprincipal ultrafilter and let $A_i \in p$, $i = 1, 2, \ldots$, with $A_{i+1} \subset A_i$ and $\bigcap A_i = \varnothing$. Let $B_i = A_i \setminus A_{i+1}$ and let $\{e_i \mid i = 1, 2, \ldots\}$ be an orthonormal system with common range projection P. Define ξ by $\xi_s = e_i$ if $s \in B_i$ and $\xi_s = 0$ if $s \notin A_1$. Then $P\xi = \xi$, $\operatorname{rank} P = \omega$, and $\xi \in \mathcal{W}_p$.

Now let $\varphi \in \alpha^\alpha$ and let $\bar{\varphi}$ denote its Stone extension. Then $\bar{\varphi}(p) = \{V \subset \alpha \mid \varphi^{-1}V \in p\} = \{\varphi U \mid U \in p\}$. Such a φ defines a map U_φ of $\mathcal{H}^\alpha$ into itself by $(U_\varphi \xi)_s = \xi_{\varphi(s)}$ and for $\xi, \eta \in \mathcal{H}^\alpha$ we have

$$\langle U_\varphi \xi, U_\varphi \eta \rangle_p = \lim_p \langle \xi_{\varphi(s)}, \eta_{\varphi(s)} \rangle = \lim_{\varphi p} \langle \xi_s, \eta_s \rangle = \langle \xi, \eta \rangle_{\varphi p}.$$

Thus φ induces an isometry of $\mathcal{H}_{\varphi p}$ into $\mathcal{H}_p$. However U_φ will in general not map $\mathcal{K}_{0,\varphi p}$, respectively $\mathcal{K}_{\kappa,\varphi p}$ into the corresponding spaces. This is only true iff there exists an $A \in p$ such that

$$(3_0) \qquad | \varphi^{-1}(\{s\}) \cap A | < \omega,$$

respectively,

$$(3_\kappa) \qquad | \varphi^{-1}(\{s\}) \cap A | \leq \kappa.$$

To see this let $\{\xi_s \mid s \in \alpha\}$ be an orthonormal basis of $\mathcal{H}$. Then $\xi \in \mathcal{W}$. Assume $U_\varphi \xi = (\xi_{\varphi s}) \in \mathcal{W} + \mathcal{N}_p$ with $U_\varphi \xi = \eta + \varsigma$, $\eta \in \mathcal{W}$, $\varsigma \in \mathcal{N}_p$. For $\varepsilon > 0$ there exists $A_\varepsilon \in p$ with $\|\varsigma_s\| < \varepsilon$ for $s \in A_\varepsilon$. Then $|\langle \eta_t \mid \xi_s \rangle| \geq 1 - \varepsilon$ for $t \in \varphi^{-1}(\{s\}) \cap A_\varepsilon$, and this set must be finite. The same proof also works if finite is replaced by κ. Conversely assume (3). For $\xi \in \mathcal{W}$ write $\eta = \chi_A U_\varphi \xi$ where χ_A is the characteristic function of A. Then $\varsigma = U_\varphi \xi - \eta \in \mathcal{N}_p$ and $\eta \in \mathcal{W}$.

Assume φ satisfies (3_0) and that U_φ maps $K_{0,\varphi p}$ onto $\mathcal{K}_{0,p}$. Choose $\xi = (\xi_s)$ such that $\{\xi_s \mid s \in \alpha\}$ is an orthonormal base of $\mathcal{H}$ and let $\eta \in \mathcal{W}$ with $U_\varphi \eta - \xi \in \mathcal{N}_p$. There exists therefore an $A \in p$ with $\|(U_\varphi \eta - \xi)_s\| < \frac{1}{2}$. Thus $\|\eta_{\varphi s} - \xi_s\| < \frac{1}{2}$, $s \in A$. Thus φ is injective on A i.e. φ is a p-equivalence.

It is obvious that the above discussion of maps $\varphi \in \alpha^\alpha$ has its counterpart in the Rudin-Keisler order of ultrafilters [**4**, §9]. The maps used to define this order should however satisfy the conditions (3_0), respectively (3_κ). Consequently we shall speak of R-K-0 or R-K-κ order, which will be defined as follows.

It is easy to see that any map φ satisfying (3_0), respectively (3_κ), is equivalent mod p to a map ψ satisfying

$$(4_0) \qquad |\psi^{-1}(s)| < \omega, \qquad s \in \alpha,$$

respectively,

$$(4_\kappa) \qquad |\psi^{-1}(s)| \leq \kappa, \qquad s \in \alpha.$$

This means there exists an $A \in p$ with $\varphi \mid A = \psi \mid A$ and thus $\bar{\varphi}(p) = \bar{\psi}(p)$. Such maps ψ form a semigroup S_0, respectively S_κ. For $p, q \in \beta\alpha$ define now $p \leq q$ if there exists a $\varphi \in S_0$, respectively $\varphi \in S_\kappa$, with $\bar{\varphi}(q) = p$ and extend this partial order to the types. The partial order obtained this way will be the R-K-0, respectively R-K-κ, order. This order is clearly weaker than the R-K order and there will be more minimal types than with respect to the R-K order. In addition we note that S_κ does not alter $\|p\|$ if $\kappa < \|p\|$, which we had assumed above. To see this assume $\|\varphi p\| < \|p\|$ and $\varphi \in S_\kappa$. Then there is a B with $|B| = \|\varphi p\|$ such that $\varphi^{-1} B = A \in p$. This however gives

$$|A| = \left| \bigcup_{s \in B} \varphi^{-1} s \right| \leq \kappa \cdot |B| < \|p\|,$$

which is impossible.

So far we have seen that a necessary condition for π_p, respectively $\pi_{k,p}$, to be irreducible is that p be minimal in the R-K-0, respectively R-K-κ, order. In order to prove the converse we have to extend Lemma 9.4 of [**5**] to our situation.

LEMMA 3. (a) *Assume $\|p\| \geq \omega$. Then p is R-K-0 minimal iff for any partition $\{d_\rho \mid \rho < \alpha, |d_\rho| < \omega\}$ of α there is an $A \in p$ with $|A \cap d_\rho| \leq 1$, $\rho < \alpha$.*

(b) *Assume $\|p\| > \kappa$. Then p is R-K-κ minimal iff for any partition $\{d_\rho \mid \rho < \alpha, |d_\rho| \leq \kappa\}$ of α there is an $A \in p$ with $|A \cap d_\rho| \leq 1$, $\rho < \alpha$.*

PROOF. (a) Assume p is R-K-0 minimal and let $\{d_\rho\}$ be such a partition of α. Define $\varphi \in \alpha^\alpha$ by $\varphi(s) = \rho$ if $s \in d_\rho$ and $d_\rho \neq \emptyset$. Then $\varphi \in S_0$ and since $\overline{\varphi p} \approx p$ there is an $A \in p$ such that $\varphi \mid A$ is one-to-one.

For the converse let $\varphi \in S_0$ and let $d_\rho = \varphi^{-1}(\rho)$. By assumption there is an $A \in p$ with $|A \cap d_\rho| \leq 1$. Then $\varphi \mid A$ is clearly one-to-one.

(b) This is shown as (a).

With this we can now show our main result.

THEOREM 1. (a) *Assume $\|p\| \geq \omega$. Then π_p is irreducible iff p is R-K-0 minimal,* $\ker \pi_p = I_{\|p\|}$. *If $\|p\| = \alpha$, then* $\dim \pi_p = 2^\alpha$.

(b) *Assume $\|p\| > \kappa$. Then $\pi_{\kappa,p}$ is irreducible iff p is R-K-κ minimal. In this case $\mathcal{W} + \mathcal{N}_p = \mathcal{W}_\kappa + \mathcal{N}_p$ and* $\ker \pi_{\kappa,p} = I_{\|p\|}$.

PROOF. (a) Assume p to be R-K-0 minimal. Our aim is to show that for any normalized vector $\xi \in \mathcal{W}$ there is an $\eta \in \mathcal{W}$ with $\eta = (\eta_s)$ such that the $\{\eta_s\}$ form an orthonormal base of $\mathcal{H}$ and such that $\xi - \eta \in \mathcal{N}_p$. In order to do this α will be decomposed into finite blocks, for which the ξ_s are almost orthogonal. We may assume $\|\xi_s\| = 1 = \|\xi\|_p$, $s \in \alpha$. Because of Lemma 1, α can be decomposed into a family $\{d_\rho \mid \rho < \alpha, |d_\rho| \leq \omega\}$ such that $\xi_s \perp \xi_t$ if $s \in d_\rho$, $t \in d_\sigma$, and $\sigma \neq \rho$. The d_ρ will now be decomposed further. To do this fix ρ and write $d_\rho = \{1, 2, 3, \dots\}$. Inductively define now $A_0 = \varnothing$, $A_1 = \{1\}$, and $A_{i+1} = \{j \mid \|P\xi_j\| \geq 1/(i+1)$ where $P = \langle \xi_l \mid l \in A_i \rangle\}$. Then $| A_i | < \omega$ and $A_i \subset A_{i+1}$. If $\bigcup A_i \neq d_\rho$, repeat the same procedure with $d_\rho \setminus \bigcup A_i$. Setting $B_{i+1} = A_{i+1} \setminus A_i$, $i = 0, 1, \dots,$ leads to a decomposition of α into $\{d_\rho\}$ and a decomposition of the infinite d_ρ into $B_{\rho,i}$ such that

(i) $\xi_s \perp \xi_t$ if $s \in d_\rho$, $t \in d_\sigma$, and $\rho \neq \sigma$,

(ii) $|\langle \xi_s, \xi_t \rangle| < 1/(i+1)$ if $s \in B_{\rho,i}$ and $t \in B_{\rho,i+k}$ for $k \geq 2$.

The selection principle, Lemma 3, allows us to find an $A \in p$ such that A intersects each of these components in at most one element. Since p is an ultrafilter we may even assume that A intersects either the $B_{\rho,2i+1}$ or the $B_{\rho,2i}$ trivially. Likewise we may assume that A intersects only the infinite or finite components nontrivially. In the latter case we are finished. In the other case we obtain a decomposition of A into sets $\{d'_\rho \mid \rho < \alpha\}$ satisfying

(i) $d'_\rho = \{(\rho, i) \mid i \in \mathbf{N}\}$,

(ii) $\xi_{(p,i)} \perp \xi_{(\sigma,i)}$, $\rho \neq \sigma$,

(iii) $|\langle \xi_{\rho,i}, \xi_{\rho,j} \rangle| < 1/(k+1)$ with $k = \min(i,j)$.

Now let $A_n = \bigcup_\rho \{(\rho, i) \mid i \leq n\}$. Then either $A_n \in p$ for some n or $A_n^c \in p$ for all n. In the first case there is a $B \in p$ intersecting each d'_ρ in at most one element. In the latter case the vector η, which one obtains from ξ by the Gram Schmidt orthogonalization of the family $\{\xi_{(\rho,i)} \mid \rho < \alpha,\ i = 1, 2, \dots\}$, satisfies $\xi - \eta \in \mathcal{N}_p$.

(b) The proof for $\xi \in \mathcal{W}_\kappa$ is similar, but much simpler.

(c) Altogether the assumptions imply that for any normalized $\xi \in \mathcal{W}$, respectively $\mathcal{W}_\kappa$, there is an $\eta \in \mathcal{W} + \mathcal{N}_p$ and a set $A \in p$ such that $\xi - \eta \in \mathcal{N}_p$ and $\langle \eta_s, \eta_t \rangle = \delta_{t,s}$ for $s, t \in A$.

Another simple manipulation of this η finally shows that we may even assume that the $\{\eta_s\}$ form an orthonormal base of $\mathcal{H}$. Thus any normalized vector $\xi \in \mathcal{K}_{0,p}$, respectively $\mathcal{K}_{\kappa,p}$ has a representing vector, whose components form an orthonormal base. The irreduciblity of π_p, respectively $\pi_{\kappa,p}$, however follows easily from this, because any two such vectors can be connected by a unitary from $B(\mathcal{H})$. The result about the dimension of π_p finally follows as in [**3**]. In fact this holds for any irreducible representation π of $B(\mathcal{H})$ with $\ker \pi = I_\alpha$.

THEOREM 2. (a) *Assume $\|p\| \geq \omega$ and assume p is R-K-0 minimal. Let $\{e_s \mid s \in \alpha\}$ be an orthonormal base of $\mathcal{H}$ and let $l^\infty(\alpha)$ be the corresponding maximal abelian *-subalgebra of diagonal operators. Then p considered as a pure state of $l^\infty(\alpha)$ has unique pure state extension*

$$\varphi_p(T) = \lim_p \langle T e_s, e_s \rangle$$

to all of $B(\mathcal{H})$.

(b) *The same result holds if finite is replaced by κ.*

PROOF. (a) For any subset $A \subset \alpha$ let $P_A = \langle e_s \mid s \in A \rangle$. If φ is any pure state extension of p and $A \in p$ then $\varphi(P_A T P_A) = \varphi(T)$ because $\varphi(P_A) = p(P_A) = 1$ [**1**]. Thus by [**2**, Theorem 2.4] we are finished if we show that for any $T \in B(\mathcal{H})$ there is an $A \in p$ such that $P_A T P_A \in l^\infty(\alpha) + \overline{\operatorname{span}}[l^\infty(\alpha), B(\mathcal{H})]$.

(b) Let $T \in B(\mathcal{H})_+$ be given. By Lemma 1 we can decompose α into countable subsets d_ρ, $0 \leq \rho < \alpha$, such that

$$\mathcal{A}e_s \perp \mathcal{A}e_t \quad \text{if } s \in d_\rho,\ t \in d_\sigma,\ \sigma \neq \rho, \tag{5}$$

holds for the C^*-algebra $\mathcal{A}$ generated by 1 and T. Let $P_\rho = P_{d_\rho}$. Now decompose the infinite d_ρ further into finite subsets by the same procedure as in [**7**] for $d = \{1, 2, 3, \dots\}$. Starting with $n_0 = 1$ construct inductively an increasing sequence of numbers n_i such that

$$\|(1 - P_{[1,n_{i+1}]})TP_{[1,n_i]}\| \leq 2^{-(i+1)}. \tag{6}$$

Let $B_{\rho,i} = [n_i + 1, n_{i+1}]$. Since p is a R-K-0 minimal ultrafilter there is an $A \in p$ such that either

$$A \cap B_{\rho,2i} = \varnothing \quad \text{and} \quad |A \cap B_{\rho,2i+1}| \leq 1$$

or

$$|A \cap B_{\rho,2i}| \leq 1 \quad \text{and} \quad A \cap B_{\rho,2i+1} = \varnothing$$

holds for all i and ρ. Assume the latter case holds.

(c) We can now write

$$P_ATP_A = \sum_\rho P_\rho P_ATP_AP_\rho = \sum P_{A\cap d_\rho}T_\rho P_{A\cap d_\rho},$$

where $T_\rho = P_\rho TP_\rho$ and it suffices to show

$$P_{A\cap d_\rho}T_\rho P_{A\cap d_\rho} \in l^\infty(d_\rho) + \overline{\text{span}}[l^\infty(d_\rho), B(l^2(d_\rho))]. \tag{7}$$

To do this fix ρ, write as before $d_\rho = \{1, 2, 3, \dots\}$, and let $A \cap B_{\rho,2i} = \{m_i\}$ if this set is nonempty. Moreover let $P_i = P_{\{m_i\}}$. Then

$$P_{A\cap d_\rho}T_\rho P_{A\cap d_\rho} = \sum_i P_iT_\rho P_i + \sum_{i\neq j} P_iT_\rho P_j = R + S.$$

Since clearly $R \in l^\infty(d_\rho)$ it suffices that S can be approximated by a linear combination of commutators. To do this we need for $i > j$

$$\|P_iT_\rho P_j\| = \|P_jT_\rho T_i\| < 2^{-(2i-1)}.$$

Then we have

$$S = \sum_{k=1}^\infty \sum_i P_iT_\rho P_{i+k} + \sum_{k=1}^\infty \sum_i P_{i+k}T_\rho P_i = \sum_{k=1}^\infty S_k + S_k^*$$

and the sum converges in norm because $\|S_k\| < 2^{-2k-2}$. Each S_k however is in the commutator of $l^\infty(d_\rho)$ with a shift of multiplicity k.

Finally we remark that our results show that $|B(\hat{\mathcal{H}})| = \alpha^{++}$ because there are α^{++} R-K-minimal α-homogeneous ultrafilters and only α^+ unitaries.

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FACHBEREICH MATHEMATIK INFORMATIK, UNIVERSITÄT OSNABRÜCK, POSTFACH 4469, D-4500 OSNABRÜCK, FEDERAL REPUBLIC OF GERMANY