A TRANSFORMATION FOR AN $n$-BALANCED $\Phi_2$

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ABSTRACT. An interesting generalization of the familiar $q$-extension of the Pfaff-Saalschiitz theorem is proved and is applied, for example, to derive a reduction formula for a certain double $q$-series. The main theorem (asserting the symmetry in $n$ and $N$ of a function $f(n,N)$ defined in terms of an $n$-balanced basic (or $q$-) hypergeometric $\Phi_2$ series by equation (8)) is essentially a $q$-extension of Sheppard's transformation.

1. Introduction and the main result. For real or complex $q$, $|q| < 1$, let

$$(\lambda; q)_\mu = \prod_{j=0}^{\infty} \left( \frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right)$$

for arbitrary $\lambda$ and $\mu$, so that

$$(\lambda; q)_l = \begin{cases} 1, & \text{if } l = 0, \\ (1 - \lambda)(1 - \lambda q) \cdots (1 - \lambda q^{l-1}), & \forall l \in \{1, 2, 3, \ldots \}, \end{cases}$$

and

$$(\lambda; q)_{\infty} = \prod_{j=0}^{\infty} (1 - \lambda q^j).$$

Denote by $\Phi_s$ a basic (or $q$-) hypergeometric series with $r$ numerator and $s$ denominator parameters (see Slater [6, Chapter 3] for details). Following Askey and Wilson [1, p. 6], we say that the $q$-hypergeometric series

$$(4) \quad \Phi_p \left[ \begin{array}{c} \alpha_0, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_p; \\ q, z \end{array} \right] = \sum_{l=0}^{\infty} \frac{(\alpha_0; q)_l \cdots (\alpha_p; q)_l}{(\beta_1; q)_l \cdots (\beta_p; q)_l} \frac{z^l}{(q; q)_l}$$

is balanced if it terminates (that is, if at least one of the numerator parameters $\alpha_0, \ldots, \alpha_p$ is of the form $q^{-N}$ ($N = 0, 1, 2, \ldots$)), if $z = q$, and if

$$(5) \quad \beta_1 \cdots \beta_p = q\alpha_0 \cdots \alpha_p,$$

it being understood, as usual, that no zeros appear in the denominator of (4). More generally, the $q$-hypergeometric series (4) is said to be $n$-balanced if this last condition (5) is replaced by

$$(6) \quad \beta_1 \cdots \beta_p = q^{n+1} \alpha_0 \cdots \alpha_p \quad (n = 0, 1, 2, \ldots),$$

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so that a zero-balanced $q$-hypergeometric series is simply called balanced.

We shall also need the Gaussian (or $q$-binomial) coefficient defined for all non-negative integers $n$ and $k$ by

$$\binom{n}{k} = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{j=1}^{k} \left( \frac{1-q^{n-j+1}}{1-q} \right), & \text{if } 1 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

The main result of the present paper is a transformation formula for an $n$-balanced $\phi_2$ series, which is contained in the following

**THEOREM.** Let $n$ and $N$ be arbitrary nonnegative integers. Then $f(n, N)$ defined in terms of an $n$-balanced $\phi_2$ series by

$$f(n, N) = \frac{(c; q)_n(c/ab; q)_N}{(c/a; q)_N(c/b; q)_N} \phi_2 \left[ \begin{array}{c} a, b, q^{-n}; \\ cq^n, abq^{1-N}/c; \\ q, q \end{array} \right]$$

is a symmetric function of $n$ and $N$.

**REMARK 1.** The theorem can be stated in its equivalent form:

$$\phi_2 \left[ \begin{array}{c} a, b, q^{-n}; \\ cq^n, abq^{1-N}/c; \\ q, q \end{array} \right] = \frac{(cq^n/a; q)_n(cq^n/b; q)_{N-n}}{(cq^n/q)_N(cq^n/ab; q)_N} \sum_{k=0}^{n} \binom{n}{k} \frac{(a; q)_k(b; q)_k(c/ab; q)_{n-k}}{(cq^n/q)_k} \left( \frac{c}{ab} \right)^k,$$

which, for $n = 0$, reduces immediately to Jackson’s sum (cf. [3, p. 111, equation (B)]; see also [6, p. 97, equation (3.3.2.2)])

$$\phi_2 \left[ \begin{array}{c} a, b, q^{-n}; \\ c, abq^{1-N}/c; \\ q, q \end{array} \right] = \frac{(c/a; q)_N(c/b; q)_N}{(c; q)_N(c/ab; q)_N}$$

for a balanced $\phi_2$ series.

**REMARK 2.** Upon replacing $a$, $b$ and $c$ by $qa$, $qb$ and $qc$, respectively, and letting $q \rightarrow 1$, formula (10) evidently yields the well-known Pfaff-Saalschiitz theorem [6, p. 49, equation (2.3.1.3)]. The general result (9), on the other hand, similarly yields a $\phi_2$ transformation which is at least as old as Sheppard [5, p. 476, equation (18)].

2. **Proof of the theorem.** Our proof of the summation formula (9) is based upon (10) and the $q$-series identity (cf. [7, p. 229, equation (6.1)]):

$$\sum_{i, m=0}^{\infty} \Omega_{l+m}(\lambda; q)_l(\mu; q)_m \frac{(\mu z)^l}{(q; q)_l} \frac{z^m}{(q; q)_m} = \sum_{n=0}^{\infty} \Omega_n(\lambda \mu; q)_n \frac{z^n}{(q; q)_n},$$

where $\{\Omega_n\}_{n=0}^{\infty}$ is a bounded sequence of complex numbers and the parameters $\lambda$ and $\mu$ are essentially arbitrary.
Denote, for convenience, the left-hand side of the summation formula (9) by \( S \). Applying the \( q \)-series identity (11) with, of course, 

\[
\Omega_l = \frac{(a;q)_l(b;q)_l}{(cq^n;q)_l(abq^{1-N}/c;q)_l}, \quad l \geq 0,
\]

we thus find that

\[
S = \sum_{l,m \geq 0} \frac{(a;q)_{l+m}(b;q)_{l+m}}{(cq^n;q)_{l+m}(abq^{1-N}/c;q)_{l+m}} \cdot \frac{(q^{-n};q)_l(q^{-N+n};q)_m q^{(1-N+n)l} q^m}{(q;q)_l(q;q)_m}
\]

\[
= \sum_{l=0}^{n} \frac{(a;q)_l(b;q)_l(q^{-n};q)_l q^{(1-N+n)l}}{(cq^n;q)_l(abq^{1-N}/c;q)_l(q;q)_l} \cdot \Phi_2 \left[ \frac{aq^l, bq^l, q^{-N+n}, \cdots}{cq^{n+l}, abq^{1-N+l}/c, q, q} \right].
\]

Summing this balanced \( \Phi_2 \) series by means of (10) with \( a, b, c \) and \( N \) replaced by \( aq^l, bq^l, cq^l \) and \( N - n \), respectively, we have

\[
S = \frac{(cq^n/a;q)_{N-n}(cq^n/b;q)_{N-n}}{(cq^n;q)_{N-n}} \sum_{l=0}^{n} \frac{(a;q)_l(b;q)_l(q^{-n};q)_l q^{(1-N+n)l}}{(cq^n;q)_l(abq^{1-N}/c;q)_l(cq^{n-l}/ab;q)_{N-n}(q;q)_l}
\]

\[
= \frac{(cq^n/a;q)_{N-n}(cq^n/b;q)_{N-n}}{(cq^n;q)_{N-n}(c/ab;q)_{N-n}} \sum_{l=0}^{n} (-1)^l q^{\frac{1}{2}l(1+2n-l)} \frac{(a;q)_l(b;q)_l(q^{-n};q)_l (c/ab)^l}{(cq^n;q)_l(q;q)_l}.
\]

From the definition (7) it is easily verified that

\[
\binom{n}{k} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = (-1)^k q^{\frac{1}{2}k(1+2n-k)} \frac{(q^{-n};q)_k}{(q;q)_k}, \quad 0 \leq k \leq n,
\]

which, when used in the last expression in (13), yields

\[
S = \frac{(cq^n/a;q)_{N-n}(cq^n/b;q)_{N-n}}{(cq^n;q)_{N-n}(c/ab;q)_{N-n}} \sum_{l=0}^{n} \frac{n}{l} \frac{(a;q)_l(b;q)_l(c/ab;q)_{n-l}}{(cq^n;q)_l} \left( \frac{c}{ab} \right)^l,
\]

thus completing the proof of the theorem.
Alternatively, as suggested by the referee, the theorem can be proven by using Sears's transformation\textsuperscript{2} [4, p. 167, equation (8.3)]

\[
\binom{4}{3}(\alpha, \beta, \gamma, q^{-m}; \lambda, \mu, \nu; q, q) = \frac{(\mu/\alpha; q)_m(\lambda\mu/\beta\gamma; q)_m}{(\mu; q)_m(\lambda\mu/\alpha\beta\gamma; q)_m} \binom{4}{3}(\alpha, \lambda/\beta, \lambda/\gamma, q^{-m}; q, q, q, q),
\]

which holds true when each $\binom{4}{3}$ series is balanced, that is, when $m$ is a nonnegative integer and (cf. equation (5)) $\lambda\mu\nu = \alpha\beta\gamma q^{1-m}$. Letting $\gamma, \nu \to 0$, (15) readily yields the transformation

\[
\binom{3}{2}(\alpha, \beta, q^{-m}; \lambda, \mu, \nu; q, q) = \frac{(\mu/\alpha; q)_m}{(\mu; q)_m} \binom{3}{2}(\lambda, \alpha q^{1-m}/\mu, \alpha q^{1-m}/\nu; q, q, q, q),
\]

between two terminating $\binom{3}{2}$ series.

Making use of (16), we observe that

\[
\binom{3}{2}(c, q)_n(c/ab, q)_n \binom{3}{2}(c/a, q)_n(c/b, q)_n (c/q)_{\infty} (c/ab, q)_{\infty} \binom{3}{2}(a, b, c q^{n}, ab q^{1-N}/c; q, q, c cq^{n-N}; q, c/ab)
\]

when $b = q^{-m} (m = 0, 1, 2, \ldots)$. Obviously, the right-hand side of (17) is symmetric in $n$ and $N$. Thus the left-hand side of (17) is symmetric in $n$ and $N$ when $b = q^{-m}$. Since both $\binom{3}{2}$ series in (17) are polynomials, (17) holds true for infinitely many values of $b$, and hence for all $b$, as long as no problems arise from division by zero.

A comparison between (17) and the definition (8) evidently completes the proof of the theorem.

3. Applications. Our assertion that $f(n, N)$ defined by (8) is a symmetric function of $n$ and $N$ might lead to a number of useful consequences.

For example, just as the Pfaff-Saalschütz theorem and its $q$-analogue (10), the general summation formula (9) has great potential for applications in various areas of combinatorial analysis (see, for details, Takács [9] and Goulden [2], and the references cited in these works). We choose to record here the following consequence of our theorem, which can indeed be proved by comparing coefficients of like powers of $z$ on both sides, using the summation formula (9):

\textsuperscript{2}See also Askey and Wilson [1, pp. 4–7] for a systematic account of Sears’s transformation (15) and of some of its numerous interesting consequences.
COROLLARY. For every bounded sequence \( \{\Omega_n\}_{n=0}^{\infty} \) of complex numbers, and for complex parameters \( \alpha, \beta, \gamma \) and \( q, |q| < 1 \),

\[
\sum_{l,m=0}^{\infty} \Omega_{l+m} \frac{(\alpha; q)_l (\beta; q)_l (\gamma/\alpha \beta; q)_m (\gamma z/\alpha \beta)^l}{(\gamma q^n; q)_l (q; q)_l (q; q)_m} z^m
\]

(18)

\[
= \frac{(\gamma; q)_n}{(\gamma/\alpha; q)_n (\gamma/\beta; q)_n} \sum_{k=0}^{n} \binom{n}{k} (\alpha; q)_k (\beta; q)_k (\gamma/\alpha \beta; q)_{n-k} \left( \frac{\gamma}{\alpha \beta} \right)^k
\]

\[
\cdot \sum_{N=0}^{\infty} \Omega_N \frac{(\gamma/\alpha; q)_N (\gamma/\beta; q)_N}{(\gamma; q)_{N+k}} \frac{z^N}{(q; q)_N},
\]

provided that each series involved converges absolutely.

REMARK 3. For \( n = 0 \), (18) yields a \( q \)-analogue of a known series identity [8, p. 313, equation (143) with \( y = 0 \)].

REFERENCES

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