PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 1, September 1987

A TRANSFORMATION FOR AN n-BALANCED ${}_3\Phi_2$

H. M. SRIVASTAVA[1]

ABSTRACT. An interesting generalization of the familiar q-extension of the Pfaff-Saalschütz theorem is proved and is applied, for example, to derive a reduction formula for a certain double q-series. The main theorem (asserting the symmetry in n and N of a function $f(n, N)$ defined in terms of an n-balanced basic (or q-) hypergeometric ${}_3\Phi_2$ series by equation (8)) is essentially a q-extension of Sheppard's transformation.

1. Introduction and the main result. For real or complex q, $|q| < 1$, let

$$(\lambda; q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right) \tag{1}$$

for arbitrary λ and μ, so that

$$(\lambda; q)_l = \begin{cases} 1, & \text{if } l = 0, \\ (1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{l-1}), & \forall l \in \{1, 2, 3, \dots\}, \end{cases} \tag{2}$$

and

$$(\lambda; q)_\infty = \prod_{j=0}^{\infty} (1 - \lambda q^j). \tag{3}$$

Denote by ${}_r\Phi_s$ a basic (or q-) hypergeometric series with r numerator and s denominator parameters (see Slater [**6**, Chapter 3] for details). Following Askey and Wilson [**1**, p. 6], we say that the q-hypergeometric series

$${}_{p+1}\Phi_p \left[\begin{matrix} \alpha_0, \dots, \alpha_p; \\ \\ \beta_1, \dots, \beta_p; \end{matrix} \; q, z \right] = \sum_{l=0}^{\infty} \frac{(\alpha_0; q)_l \cdots (\alpha_p; q)_l}{(\beta_1; q)_l \cdots (\beta_p; q)_l} \frac{z^l}{(q; q)_l} \tag{4}$$

is *balanced* if it terminates (that is, if at least one of the numerator parameters $\alpha_0, \dots, \alpha_p$ is of the form q^{-N} $(N = 0, 1, 2, \dots)$), if $z = q$, and if

$$\beta_1 \cdots \beta_p = q\alpha_0 \cdots \alpha_p, \tag{5}$$

it being understood, as usual, that no zeros appear in the denominator of (4). More generally, the q-hypergeometric series (4) is said to be *n-balanced* if this last condition (5) is replaced by

$$\beta_1 \cdots \beta_p = q^{n+1} \alpha_0 \cdots \alpha_p \qquad (n = 0, 1, 2, \dots), \tag{6}$$

Received by the editors March 17, 1986 and, in revised form, May 29, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 33A30; Secondary 05A30, 11B65.

Key words and phrases. Pfaff-Saalschütz theorem, basic (or q-) hypergeometric series, n-balanced series, Sheppard's transformation, Gaussian (or q-binomial) coefficient, Jackson's sum, q-series identity, Sears's transformation, combinatorial analysis.

[1] This work was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant A-7353.

so that a zero-balanced q-hypergeometric series is simply called balanced.

We shall also need the Gaussian (or q-binomial) coefficient defined for all nonnegative integers n and k by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{j=1}^{k} \left(\frac{1-q^{n-j+1}}{1-q^j} \right), & \text{if } 1 \le k \le n, \\ 0, & \text{if } k > n. \end{cases} \tag{7}$$

The main result of the present paper is a transformation formula for an n-balanced ${}_3\Phi_2$ series, which is contained in the following

THEOREM. *Let n and N be arbitrary nonnegative integers. Then $f(n,N)$ defined in terms of an n-balanced ${}_3\Phi_2$ series by*

$$f(n,N) = \frac{(c;q)_N (c/ab;q)_N}{(c/a;q)_N (c/b;q)_N} \, {}_3\Phi_2 \left[\begin{matrix} a, b, q^{-N}; \\ cq^n, abq^{1-N}/c; \end{matrix} \; q, q \right] \tag{8}$$

is a symmetric function of n and N.

REMARK 1. The theorem can be stated in its *equivalent* form:

$$\begin{aligned} {}_3\Phi_2 \left[\begin{matrix} a, b, q^{-N}; \\ cq^n, abq^{1-N}/c; \end{matrix} \; q, q \right] &= \frac{(cq^n/a;q)_{N-n} (cq^n/b;q)_{N-n}}{(cq^n;q)_{N-n} (c/ab;q)_N} \\ &\quad \cdot \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a;q)_k (b;q)_k (c/ab;q)_{n-k}}{(cq^N;q)_k} \left(\frac{c}{ab} \right)^k, \end{aligned} \tag{9}$$

which, for $n = 0$, reduces immediately to Jackson's sum (cf. [**3**, p. 111, equation (B)]; see also [**6**, p. 97, equation (3.3.2.2)])

$${}_3\Phi_2 \left[\begin{matrix} a, b, q^{-N}; \\ c, abq^{1-N}/c; \end{matrix} \; q, q \right] = \frac{(c/a;q)_N (c/b;q)_N}{(c;q)_N (c/ab;q)_N} \tag{10}$$

for a balanced ${}_3\Phi_2$ series.

REMARK 2. Upon replacing a, b and c by q^a, q^b and q^c, respectively, and letting $q \to 1$, formula (10) evidently yields the well-known Pfaff-Saalschütz theorem [**6**, p. 49, equation (2.3.1.3)]. The general result (9), on the other hand, similarly yields a ${}_3F_2$ transformation which is at least as old as Sheppard [**5**, p. 476, equation (18)].

2. Proof of the theorem. Our proof of the summation formula (9) is based upon (10) and the q-series identity (cf. [**7**, p. 229. equation (6.1)]):

$$\begin{aligned} &\sum_{l,m=0}^{\infty} \Omega_{l+m} (\lambda;q)_l (\mu;q)_m \frac{(\mu z)^l}{(q;q)_l} \frac{z^m}{(q;q)_m} \\ &\qquad = \sum_{n=0}^{\infty} \Omega_n (\lambda\mu;q)_n \frac{z^n}{(q;q)_n}, \end{aligned} \tag{11}$$

where $\{\Omega_n\}_{n=0}^{\infty}$ is a bounded sequence of complex numbers and the parameters λ and μ are essentially arbitrary.

Denote, for convenience, the left-hand side of the summation formula (9) by S. Applying the q-series identity (11) with, of course,

$$
\begin{aligned}
&z = q, \quad \lambda = q^{-n}, \quad \mu = q^{-N+n}, \\
&\Omega_l = \frac{(a;q)_l (b;q)_l}{(cq^n;q)_l (abq^{1-N}/c;q)_l}, \qquad l \geq 0,
\end{aligned} \tag{12}
$$

we thus find that

$$
\begin{aligned}
S = &\sum_{l,m\geq 0} \frac{(a;q)_{l+m}(b;q)_{l+m}}{(cq^n;q)_{l+m}(abq^{1-N}/c;q)_{l+m}} \\
&\cdot (q^{-n};q)_l (q^{-N+n};q)_m \frac{q^{(1-N+n)l}}{(q;q)_l} \frac{q^m}{(q;q)_m} \\
= &\sum_{l=0}^{n} \frac{(a;q)_l (b;q)_l (q^{-n};q)_l}{(cq^n;q)_l (abq^{1-N}/c;q)_l} \frac{q^{(1-N+n)l}}{(q;q)_l} \\
&\cdot {}_3\Phi_2 \left[\begin{matrix} aq^l, bq^l, q^{-N+n}; \\ cq^{n+l}, abq^{1-N+l}/c; \end{matrix} \; q, q \right].
\end{aligned}
$$

Summing this balanced ${}_3\Phi_2$ series by means of (10) with a, b, c and N replaced by aq^l, bq^l, cq^l and $N-n$, respectively, we have

$$
\begin{aligned}
S = &\frac{(cq^n/a;q)_{N-n}(cq^n/b;q)_{N-n}}{(cq^n;q)_{N-n}} \\
&\cdot \sum_{l=0}^{n} \frac{(a;q)_l (b;q)_l (q^{-n};q)_l}{(cq^N;q)_l (abq^{1-N}/c;q)_l (cq^{n-l}/ab;q)_{N-n}} \frac{q^{(1-N+n)l}}{(q;q)_l} \\
= &\frac{(cq^n/a;q)_{N-n}(cq^n/b;q)_{N-n}}{(cq^n;q)_{N-n}(c/ab;q)_N} \\
&\cdot \sum_{l=0}^{n} (-1)^l q^{\frac{1}{2}l(1+2n-l)} \frac{(a;q)_l (b;q)_l (q^{-n};q)_l}{(cq^N;q)_l} \frac{(c/ab)^l}{(q;q)_l}.
\end{aligned} \tag{13}
$$

From the definition (7) it is easily verified that

$$
\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = (-1)^k q^{\frac{1}{2}k(1+2n-k)} \frac{(q^{-n};q)_k}{(q;q)_k}, \qquad 0 \leq k \leq n, \tag{14}
$$

which, when used in the last expression in (13), yields

$$
S = \frac{(cq^n/a;q)_{N-n}(cq^n/b;q)_{N-n}}{(cq^n;q)_{N-n}(c/ab;q)_N} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} \frac{(a;q)_l (b;q)_l (c/ab;q)_{n-l}}{(cq^N;q)_l} \left(\frac{c}{ab}\right)^l,
$$

thus completing the proof of the theorem.

Alternatively, as suggested by the referee, the theorem can be proven by using Sears's transformation[2] [**4**, p. 167, equation (8.3)]

$$
\begin{aligned}
{}_4\Phi_3\left[\begin{matrix}\alpha,\beta,\gamma,q^{-m};\\ \\ \lambda,\mu,\nu;\end{matrix}\; q,q\right] &= \frac{(\mu/\alpha;q)_m(\lambda\mu/\beta\gamma;q)_m}{(\mu;q)_m(\lambda\mu/\alpha\beta\gamma;q)_m} \\
&\quad\cdot {}_4\Phi_3\left[\begin{matrix}\alpha,\lambda/\beta,\lambda/\gamma,q^{-m};\\ \\ \lambda,\alpha q^{1-m}/\mu,\alpha q^{1-m}/\nu;\end{matrix}\; q,q\right],
\end{aligned}
\tag{15}
$$

which holds true when each ${}_4\Phi_3$ series is balanced, that is, when m is a nonnegative integer and (cf. equation (5)) $\lambda\mu\nu = \alpha\beta\gamma q^{1-m}$. Letting $\gamma,\nu \to 0$, (15) readily yields the transformation

$$
{}_3\Phi_2\left[\begin{matrix}\alpha,\beta,q^{-m};\\ \\ \lambda,\mu;\end{matrix}\; q,q\right] = \frac{(\mu/\alpha;q)_m}{(\mu;q)_m}\alpha^m\; {}_3\Phi_2\left[\begin{matrix}\alpha,\lambda/\beta,q^{-m};\\ \\ \lambda,\alpha q^{1-m}/\mu;\end{matrix}\; q,\frac{\beta q}{\mu}\right]
\tag{16}
$$

between two terminating ${}_3\Phi_2$ series.

Making use of (16), we observe that

$$
\begin{aligned}
&\frac{(c;q)_N(c/ab;q)_N}{(c/a;q)_N(c/b;q)_N}\,{}_3\Phi_2\left[\begin{matrix}a,b,q^{-N};\\ \\ cq^n,abq^{1-N}/c;\end{matrix}\; q,q\right] \\
&\quad= \frac{(c;q)_\infty(c/ab;q)_\infty}{(c/a;q)_\infty(c/b;q)_\infty}\,{}_3\Phi_2\left[\begin{matrix}a,b,cq^{n+N};\\ \\ cq^n,cq^N;\end{matrix}\; q,\frac{c}{ab}\right]
\end{aligned}
\tag{17}
$$

when $b = q^{-m}$ $(m = 0,1,2,\dots)$. Obviously, the right-hand side of (17) is symmetric in n and N. Thus the left-hand side of (17) is symmetric in n and N when $b = q^{-m}$. Since both ${}_3\Phi_2$ series in (17) are polynomials, (17) holds true for infinitely many values of b, and hence for all b, as long as no problems arise from division by zero.

A comparison between (17) and the definition (8) evidently completes the proof of the theorem.

3. Applications. Our assertion that $f(n,N)$ defined by (8) is a symmetric function of n and N might lead to a number of useful consequences.

For example, just as the Pfaff-Saalschütz theorem and its q-analogue (10), the general summation formula (9) has great potential for applications in various areas of combinatorial analysis (see, for details, Takács [**9**] and Goulden [**2**], and the references cited in these works). We choose to record here the following consequence of our theorem, which can indeed be proved by comparing coefficients of like powers of z on both sides, using the summation formula (9):

[2]See also Askey and Wilson [**1**, pp. 4–7] for a systematic account of Sears's transformation (15) and of some of its numerous interesting consequences.

COROLLARY. *For every bounded sequence* $\{\Omega_n\}_{n=0}^{\infty}$ *of complex numbers, and for complex parameters* α, β, γ *and* q, $|q| < 1$,

$$\sum_{l,m=0}^{\infty} \Omega_{l+m} \frac{(\alpha;q)_l(\beta;q)_l(\gamma/\alpha\beta;q)_m}{(\gamma q^n;q)_l} \frac{(\gamma z/\alpha\beta)^l}{(q;q)_l} \frac{z^m}{(q;q)_m}$$

$$= \frac{(\gamma;q)_n}{(\gamma/\alpha;q)_n(\gamma/\beta;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\alpha;q)_k(\beta;q)_k(\gamma/\alpha\beta;q)_{n-k} \left(\frac{\gamma}{\alpha\beta}\right)^k \tag{18}$$

$$\cdot \sum_{N=0}^{\infty} \Omega_N \frac{(\gamma/\alpha;q)_N(\gamma/\beta;q)_N}{(\gamma;q)_{N+k}} \frac{z^N}{(q;q)_N},$$

provided that each series involved converges absolutely.

REMARK 3. For $n = 0$, (18) yields a q-analogue of a known series identity [**8**, p. 313, equation (143) *with* $y = 0$].

REFERENCES

1. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc., Vol. 54, No. 319 (1985).
2. I. P. Goulden, *A bijective proof of the q-Saalschütz theorem*, Discrete Math. **57** (1985), 39–44.
3. F. H. Jackson, *Summation of q-hypergeometric series*, Messenger Math. **50** (1921), 101–112.
4. D. B. Sears, *On the transformation theory of basic hypergeometric functions*, Proc. London Math. Soc. (2) **53** (1951), 158–180.
5. W. F. Sheppard, *Summation of the coefficients of some terminating hypergeometric series*, Proc. London Math. Soc. (2) **10** (1912), 469–478.
6. L. J. Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, Cambridge, London and New York, 1966.
7. H. M. Srivastava and V. K. Jain, *q-Series identities and reducibility of basic double hypergeometric functions*, Canad. J. Math. **38** (1986), 215–231.
8. H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian hypergeometric series*, Halsted Press (Ellis Horwood Limited, Chichester); Wiley, New York, Chichester, Brisbane, and Toronto, 1985.
9. L. Takács, *On an identity of Shih-Chieh Chu*, Acta Sci. Math. (Szeged) **34** (1973), 383–391.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VICTORIA, VICTORIA, BRITISH COLUMBIA V8W 2Y2, CANADA