A TRANSFORMATION FOR AN \( n \)-BALANCED \( 3 \Phi_2 \)

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ABSTRACT. An interesting generalization of the familiar \( q \)-extension of the Pfaff-Saalschiitz theorem is proved and is applied, for example, to derive a reduction formula for a certain double \( q \)-series. The main theorem (asserting the symmetry in \( n \) and \( N \) of a function \( f(n, N) \) defined in terms of an \( n \)-balanced basic (or \( q \)-) hypergeometric \( 3 \Phi_2 \) series by equation (8)) is essentially a \( q \)-extension of Sheppard's transformation.

1. Introduction and the main result. For real or complex \( q \), \( |q| < 1 \), let

\[
(\lambda; q)_\mu = \prod_{j=0}^{\infty} \left( \frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right)
\]

for arbitrary \( \lambda \) and \( \mu \), so that

\[
(\lambda; q)_l = \begin{cases} 
1, & \text{if } l = 0, \\
(1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{l-1}), & \forall l \in \{1, 2, 3, \ldots\},
\end{cases}
\]

and

\[
(\lambda; q)_\infty = \prod_{j=0}^{\infty} (1 - \lambda q^j).
\]

Denote by \( \Phi_r \) a basic (or \( q \)-) hypergeometric series with \( r \) numerator and \( s \) denominator parameters (see Slater [6, Chapter 3] for details). Following Askey and Wilson [1, p. 6], we say that the \( q \)-hypergeometric series

\[
\Phi_{p+1} \left[ \begin{array}{c}
\alpha_0, \ldots, \alpha_p; \\
\beta_1, \ldots, \beta_p;
\end{array} q, z \right] = \sum_{l=0}^{\infty} \frac{(\alpha_0; q)_l \cdots (\alpha_p; q)_l \ z^l}{(\beta_1; q)_l \cdots (\beta_p; q)_l (q; q)_l}
\]

is balanced if it terminates (that is, if at least one of the numerator parameters \( \alpha_0, \ldots, \alpha_p \) is of the form \( q^{-N} \) \( (N = 0, 1, 2, \ldots) \)), if \( z = q \), and if

\[
\beta_1 \cdots \beta_p = q \alpha_0 \cdots \alpha_p,
\]

it being understood, as usual, that no zeros appear in the denominator of (4). More generally, the \( q \)-hypergeometric series (4) is said to be \( n \)-balanced if this last condition (5) is replaced by

\[
\beta_1 \cdots \beta_p = q^{n+1} \alpha_0 \cdots \alpha_p \quad (n = 0, 1, 2, \ldots),
\]

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so that a zero-balanced \( q \)-hypergeometric series is simply called balanced.

We shall also need the Gaussian (or \( q \)-binomial) coefficient defined for all non-negative integers \( n \) and \( k \) by

\[
\binom{n}{k} = \begin{cases} 
1, & \text{if } k = 0, \\
\prod_{j=1}^{k} \left( \frac{1-q^{n-j+1}}{1-q} \right), & \text{if } 1 \leq k \leq n, \\
0, & \text{if } k > n.
\end{cases}
\]

The main result of the present paper is a transformation formula for an \( n \)-balanced \( {}_3\Phi_2 \) series, which is contained in the following

**THEOREM.** Let \( n \) and \( N \) be arbitrary nonnegative integers. Then \( f(n, N) \) defined in terms of an \( n \)-balanced \( {}_3\Phi_2 \) series by

\[
f(n, N) = \frac{(c; q)_n (c/ab; q)_N}{(c/a; q)_N (c/b; q)_N} \begin{bmatrix} a, b, q^{-N}; \\
cq^n, abq^{1-N}/c; \end{bmatrix}
\]

is a symmetric function of \( n \) and \( N \).

**REMARK 1.** The theorem can be stated in its equivalent form:

\[
\begin{align*}
\sum_{k=0}^{n} & \binom{n}{k} (a; q)_k (b; q)_k (c/ab; q)_{N-k} \left( \frac{c}{ab} \right)^k \\
& \times (cq^n; q)_k (c/ab; q)_N \\
& = \frac{(cq^n/a; q)_N (cq^n/b; q)_{N-n}}{(cq^n; q)_N (c/ab; q)_N} \begin{bmatrix} a, b, q^{-N}; \\
cq^n, abq^{1-N}/c; \end{bmatrix}
\end{align*}
\]

which, for \( n = 0 \), reduces immediately to Jackson's sum (cf. [3, p. 111, equation (B)]; see also [6, p. 97, equation (3.3.2.2)])

\[
\begin{align*}
\sum_{k=0}^{n} & \binom{n}{k} (a; q)_k (b; q)_k (c/ab; q)_{N-k} \left( \frac{c}{ab} \right)^k \\
& \times (cq^n; q)_k (c/ab; q)_N \\
& = \frac{(c/a; q)_N (c/b; q)_N}{(c; q)_N (c/ab; q)_N} \\
\end{align*}
\]

for a balanced \( {}_3\Phi_2 \) series.

**REMARK 2.** Upon replacing \( a, b \) and \( c \) by \( qa, qb \) and \( qc \), respectively, and letting \( q \rightarrow 1 \), formula (10) evidently yields the well-known Pfaff-Saalschütz theorem [6, p. 49, equation (2.3.1.3)]. The general result (9), on the other hand, similarly yields a \( {}_3F_2 \) transformation which is at least as old as Sheppard [5, p. 476, equation (18)].

**2. Proof of the theorem.** Our proof of the summation formula (9) is based upon (10) and the \( q \)-series identity (cf. [7, p. 229, equation (6.1)]):

\[
\sum_{l,m=0}^{\infty} \Omega_{l+m}(\lambda; q)_l(\mu; q)_m \left( \frac{\mu z}{q} \right)^l \frac{z^m}{(q; q)_l (q; q)_m} \\
= \sum_{n=0}^{\infty} \Omega_n(\lambda \mu; q)_n \frac{z^n}{(q; q)_n},
\]

where \( \{\Omega_n\}_{n=0}^{\infty} \) is a bounded sequence of complex numbers and the parameters \( \lambda \) and \( \mu \) are essentially arbitrary.
Denote, for convenience, the left-hand side of the summation formula (9) by $S$. Applying the $q$-series identity (11) with, of course,

$$z = q, \quad \lambda = q^{-n}, \quad \mu = q^{-N+n},$$

$$\Omega_l = \frac{(a;q)_l(b;q)_l}{(cq^n;q)_l(abq^{1-N}/c;q)_l}, \quad l \geq 0,$$

we thus find that

$$S = \sum_{l,m \geq 0} \frac{(a;q)_{l+m}(b;q)_{l+m}}{(cq^n;q)_{l+m}(abq^{1-N}/c;q)_{l+m}} \cdot (q^{-n};q)_l(q^{-N+n};q)_m \frac{q^{(1-N+n)l}}{(q;q)_l} \frac{q^m}{(q;q)_m}$$

$$= \sum_{l=0}^{n} \frac{(a;q)_l(b;q)_l(q^{-n};q)_l}{(cq^n;q)_l(abq^{1-N}/c;q)_l} \frac{q^{(1-N+n)l}}{(q;q)_l} \cdot \Phi_2 \left[ \begin{array}{c} aq^l, bq^l, q^{-N+n} \\ cq^{n+l}, abq^{1-N+l}/c; q \end{array} \right].$$

Summing this balanced $\Phi_2$ series by means of (10) with $a, b, c$ and $N$ replaced by $aq^l, bq^l, cq^l$ and $N - n$, respectively, we have

$$S = \frac{(cq^n/a;q)_{N-n}(cq^n/b;q)_{N-n}}{(cq^n;q)_{N-n}}$$

$$\cdot \sum_{l=0}^{n} \frac{(a;q)_l(b;q)_l(q^{-n};q)_l}{(cq^N;q)_l(abq^{1-N}/c;q)_l(cq^{n-l}/ab;q)_{N-n}} \frac{q^{(1-N+n)l}}{(q;q)_l}$$

$$= \frac{(cq^n/a;q)_{N-n}(cq^n/b;q)_{N-n}}{(cq^n;q)_{N-n}(c/ab;q)_N}$$

$$\cdot \sum_{l=0}^{n} (-1)^l q^{\frac{1}{2}l(1+2n-l)} \frac{(a;q)_l(b;q)_l(q^{-n};q)_l}{(cq^N;q)_l} \frac{(c/ab)^l}{(q;q)_l}.$$

From the definition (7) it is easily verified that

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = (-1)^k q^{\frac{1}{2}k(1+2n-k)} \frac{q^{-n};q)_k}{(q;q)_k}, \quad 0 \leq k \leq n,$$

which, when used in the last expression in (13), yields

$$S = \frac{(cq^n/a;q)_{N-n}(cq^n/b;q)_{N-n}}{(cq^n;q)_{N-n}(c/ab;q)_N} \sum_{l=0}^{n} \left[ \begin{array}{c} n \\ l \end{array} \right] \frac{(a;q)_l(b;q)_l(c/ab;q)_{n-l}}{(cq^N;q)_l} \left( \frac{c}{ab} \right)^l,$$

thus completing the proof of the theorem.
Alternatively, as suggested by the referee, the theorem can be proven by using Sears’s transformation \(^2\) [4, p. 167, equation (8.3)]

\[
\Phi_3^4 \left[ \begin{array}{c}
\alpha, \beta, \gamma, q^{-m}; \\
\lambda, \mu, \nu;
\end{array} \left| \begin{array}{c}
q, q
\end{array} \right. \right] = \frac{(\mu/\alpha; q)_m (\lambda\mu/\beta\gamma; q)_m}{(\mu; q)_m (\lambda\mu/\alpha\beta\gamma; q)_m} \Phi_3^4 \left[ \begin{array}{c}
\alpha/\lambda, \lambda/\beta, \lambda/\gamma, q^{-m}; \\
\lambda, \alpha q^{1-m}/\mu, \alpha q^{1-m}/\nu;
\end{array} \left| \begin{array}{c}
q, q
\end{array} \right. \right],
\]

which holds true when each \(\Phi_3^4\) series is balanced, that is, when \(m\) is a nonnegative integer and (cf. equation (5)) \(\lambda\mu\nu = \alpha\beta\gamma q^{1-m}\). Letting \(\gamma, \nu \to 0\), (15) readily yields the transformation

\[
\Phi_2^3 \left[ \begin{array}{c}
\alpha, \beta, q^{-m}; \\
\lambda, \mu;
\end{array} \left| \begin{array}{c}
q, q
\end{array} \right. \right] = \frac{(\mu/\alpha; q)_m}{(\mu; q)_m} \alpha^m \Phi_2^3 \left[ \begin{array}{c}
\alpha/\lambda, \lambda/\beta, q^{-m}; \\
\lambda, \alpha q^{1-m}/\mu;
\end{array} \left| \begin{array}{c}
\beta q \mu
\end{array} \right. \right]
\]

between two terminating \(\Phi_2^3\) series.

Making use of (16), we observe that

\[
\Phi_2^3 \left[ \begin{array}{c}
(c; q)_N (c/ab; q)_N \\
(c/a; q)_N (c/b; q)_N
\end{array} \left| \begin{array}{c}
a, b, q^{-N}; \\
aq^n, abq^{1-N}/c;
\end{array} \right. \right] = \frac{(c; q)_\infty (c/ab; q)_\infty}{(c/a; q)_\infty (c/b; q)_\infty} \Phi_2^3 \left[ \begin{array}{c}
a, b, cq^{n+N}; \\
cq^n, cq^n;
\end{array} \left| \begin{array}{c}
q, c/ab
\end{array} \right. \right]
\]

when \(b = q^{-m} (m = 0, 1, 2, \ldots)\). Obviously, the right-hand side of (17) is symmetric in \(n\) and \(N\). Thus the left-hand side of (17) is symmetric in \(n\) and \(N\) when \(b = q^{-m}\). Since both \(\Phi_2^3\) series in (17) are polynomials, (17) holds true for infinitely many values of \(b\), and hence for all \(b\), as long as no problems arise from division by zero.

A comparison between (17) and the definition (8) evidently completes the proof of the theorem.

3. Applications. Our assertion that \(f(n, N)\) defined by (8) is a symmetric function of \(n\) and \(N\) might lead to a number of useful consequences.

For example, just as the Pfaff-Saalschütz theorem and its \(q\)-analogue (10), the general summation formula (9) has great potential for applications in various areas of combinatorial analysis (see, for details, Takács [9] and Goulden [2], and the references cited in these works). We choose to record here the following consequence of our theorem, which can indeed be proved by comparing coefficients of like powers of \(z\) on both sides, using the summation formula (9):

\[^2\text{See also Askey and Wilson [1, pp. 4–7] for a systematic account of Sears’s transformation (15) and of some of its numerous interesting consequences.}\]
COROLLARY. For every bounded sequence \( \{\Omega_n\}_{n=0}^{\infty} \) of complex numbers, and for complex parameters \( \alpha, \beta, \gamma \) and \( q, |q| < 1 \),

\[
\sum_{l,m=0}^{\infty} \Omega_{l+m} \frac{(\alpha;q)_l(\beta;q)_l(\gamma/\alpha\beta;q)_m}{(\gamma^q;q)_l} \frac{(\gamma^q/\alpha\beta)_l}{(q;q)_l} \frac{z^m}{(q;q)_m} = \frac{(\gamma;q)_n}{(\gamma/\alpha;q)_n(\gamma/\beta;q)_n} \sum_{k=0}^{n} \binom{n}{k} (\alpha;q)_k(\beta;q)_k(\gamma/\alpha\beta;q)_{n-k} \left( \frac{\gamma}{\alpha\beta} \right)^k 
\]

(18)

\[
\sum_{N=0}^{\infty} \Omega_N \frac{(\gamma/\alpha;q)_N(\gamma/\beta;q)_N}{(\gamma;q)_N} \frac{z^N}{(q;q)_N},
\]

provided that each series involved converges absolutely.

REMARK 3. For \( n = 0 \), (18) yields a \( q \)-analogue of a known series identity \[8, p. 313, equation (143) with \( y = 0 \)].

REFERENCES

3. F. H. Jackson, Summation of \( q \)-hypergeometric series, Messenger Math. 50 (1921), 101–112.