PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 1, September 1987

BOUNDARY LIMITS OF GREEN POTENTIALS OF GENERAL ORDER

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ABSTRACT. Our aim in this note is to discuss the boundary limits of Green potentials in a half-space of the n-dimensional euclidean space. As a corollary, our result below includes the recent result of Stoll [**5**] concerning the boundary limits of superharmonic functions along curves tending to the boundary.

In the half-space $D = \{x = (x_1, \dots, x_n); x_n > 0\}$, $n \geq 2$, let G_α be the Green function of order α, that is,

$$G_\alpha(x,y) = \begin{cases} |x-y|^{\alpha-n} - |\bar{x}-y|^{\alpha-n} & \text{if } 0 < \alpha < n, \\ \log(|\bar{x}-y|/|x-y|) & \text{if } \alpha = n, \end{cases}$$

where $\bar{x} = (x_1, \dots, x_{n-1}, -x_n)$ for $x = (x_1, \dots, x_{n-1}, x_n)$. For a nonnegative (Radon) measure μ on D, we define

$$G_\alpha\mu(x) = \int_D G_\alpha(x,y)\, d\mu(y).$$

Then it is noted (see, e.g. [**4**, Lemma 1]) that $G_\alpha\mu \not\equiv \infty$ if and only if

$$\int_D (1+|y|)^{\alpha-n-2} y_n \, d\mu(y) < \infty. \tag{1}$$

Following Fuglede [**2**], we define an outer capacity C_{G_α} by

$$C_{G_\alpha}(E) = \inf \mu(D), \qquad E \subset D,$$

where the infimum is taken over all nonnegative measures μ on D such that $G_\alpha\mu(x) \geq 1$ for every $x \in E$. In view of [**2**, Théorème 7.8], for any Borel subset E of D, $C_{G_\alpha}(E)$ is equal to $\sup \nu(D)$, where the supremum is taken over all nonnegative measures ν supported by E such that $G_\alpha\nu(x) \leq 1$ for every $x \in D$.

Following Aikawa [**1**], we say that a set E in D is α-thin at the boundary ∂D if

$$\sum_{j=1}^{\infty} 2^{j(n-\alpha)} C_{G_\alpha}(E_j) < \infty,$$

where $E_j = \{x = (x_1, \dots, x_n) \in E; 2^{-j} \leq x_n < 2^{-j+1}\}$.

Our aim in this note is to establish the following result.

Received by the editors October 21, 1985 and, in revised form, June 20, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 31B15, 31B25.
Key words and phrases. Green potentials, thin sets, capacity.

THEOREM. *Let μ be a nonnegative measure on D satisfying* (1). *Then there exists a set $E \subset D$, which is α-thin at ∂D, such that*

$$\lim_{x_n \downarrow 0, x \in D'-E} x_n^{n-\alpha+1} G_\alpha \mu(x) = 0$$

for any bounded open set $D' \subset D$.

COROLLARY. *Let $n = \alpha = 2$ and let μ be a nonnegative measure on D satisfying* (1). *If $\gamma(t)$, $0 \le t < 1$, is a bounded curve in D for which there is a sequence $\{t_j\}$ such that $\gamma(t_j)$ tends to the boundary, then* $\liminf_{t\to 1} \gamma_2(t) G_2\mu(\gamma(t)) = 0$, *where* $\gamma(t) = (\gamma_1(t), \gamma_2(t))$.

This corollary was first proved by Stoll [**5**] for Green potentials in a unit disc, and it is an easy consequence of our theorem if one notes that γ is not 2-thin at ∂D (see Lemma 4 below).

PROOF OF THE THEOREM. Write $G_\alpha \mu = u_1 + u_2$, where

$$u_1(x) = \int_{B(x, x_n/2)} G_\alpha(x, y)\, d\mu(y),$$

$$u_2(x) = \int_{D - B(x, x_n/2)} G_\alpha(x, y)\, d\mu(y);$$

here $B(x, r)$ denotes the open ball with center at x and radius r. We first prove that $G_\alpha(x, y) \le M x_n y_n |x-y|^{\alpha-n-2}$ for any x and y in D, where M is a positive constant. We shall give a proof only in the case $\alpha < n$. Let $t = |\bar{x} - y|/|x - y|$ and write $G_\alpha(x, y) = |\bar{x}-y|^{\alpha-n}(t^{n-\alpha}-1)$. Find $M' > 0$ such that $t^{n-\alpha}-1 < M'(t-1)t^{n-\alpha-1}$ whenever $t > 1$. Since $t - 1 = 4x_n y_n |x - y|^{-1}(|\bar{x} - y| + |x - y|)^{-1}$, we derive the required inequality. Let D' be a bounded open set in D. By the inequality we have

$$x_n^{n-\alpha+1} G_\alpha(x, y) \le M x_n^{n-\alpha+2} |x - y|^{\alpha-n-2} y_n \le \mathrm{const}(1 + |y|)^{\alpha-n-2} y_n$$

whenever $x \in D'$ and $y \in D - B(x, x_n/2)$. Since $\lim_{x_n \to 0} x_n^{n-\alpha+1} G_\alpha(x, y) = 0$ for fixed $y \in D$, we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x_n \to 0, x \in D'} x_n^{n-\alpha+1} u_2(x) = 0.$$

By assumption (1) we can find a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j \int_{\{y \in D; 2^{-j-1} < y_n < 2^{-j+2}\}} (1 + |y|)^{\alpha-n-2} y_n \, d\mu(y) < \infty. \tag{2}$$

Consider the sets

$$E_j = \{x \in D; 2^{-j} \le x_n < 2^{-j+1}, x_n^{n-\alpha+1} u_1(x) \ge a_j^{-1}\}$$

and $E = \bigcup_{j=1}^{\infty} E_j$. For a bounded open set $D' \subset D$, set

$$D'_j = \bigcup_{x \in E_j \cap D'} B(x, x_n/2).$$

Then $D'_j \subset \{y \in D; 2^{-j-1} < y_n < 2^{-j+2}\}$. If $x \in E_j \cap D'$, then

$$a_j^{-1} \le x_n^{n-\alpha+1} u_1(x) \le 2 \cdot 2^{(-j+1)(n-\alpha)} \int_{D'_j} G_\alpha(x, y) y_n \, d\mu(y),$$

so that

$$C_{G_\alpha}(E_j \cap D') \le 2a_j 2^{(-j+1)(n-\alpha)} \int_{D'_j} y_n \, d\mu(y).$$

Hence, since $\sum_{j=1}^{\infty} a_j \int_{D'_j} y_n \, d\mu(y) < \infty$ by (2), it follows that $E \cap D'$ is α-thin at ∂D. Clearly,

$$\lim_{x_n \to 0, x \in D'-E} x_n^{n-\alpha+1} u_1(x) = 0.$$

We now find a sequence $\{m(j)\}$ of positive integers such that

$$F = \bigcup_{k=1}^{\infty} \left(\bigcup_{j=m(k)}^{\infty} E_j \cap B(0,k) \right)$$

is α-thin at ∂D. Then F satisfies all the conditions required on E in the theorem, and hence the proof of the theorem is completed.

To prove the corollary, we prepare several lemmas.

LEMMA 1. *For $r > 0$ and $E \subset D$, set $rE = \{rx; x \in E\}$. Then*

$$C_{G_\alpha}(rE) = r^{n-\alpha} C_{G_\alpha}(E).$$

This lemma follows readily from the fact that $G_\alpha(rx, ry) = r^{\alpha-n} G_\alpha(x,y)$ whenever $r > 0$ and $x, y \in D$.

For a set E in the upper half-plane D, denote by E^* the projection of E to the half-line $l = \{(0, x_2); x_2 > 0\}$.

LEMMA 2. *For a set E in the upper half-plane, we have*

$$C_{G_\alpha}(E^*) \le C_{G_\alpha}(E).$$

PROOF. For a point $z = (z_1, z_2)$, let $z^* = (0, z_2)$. Since $G_\alpha(x,y) \le G_\alpha(x^*, y^*)$ whenever $x, y \in D$, we obtain

$$G_\alpha \mu(x) \le \int G_\alpha(x^*, y^*) \, d\mu(y) \le G_\alpha \mu^*(x^*)$$

for a nonnegative measure μ on D, where $\mu^*(A) = \mu(\{x; x^* \in A\})$ for a Borel set $A \subset l$. Thus the required assertion follows readily from the definition of capacity C_{G_α}.

LEMMA 3. *If $1 < \alpha \le 2$, then l is not α-thin at ∂D.*

PROOF. For a nonnegative integer j, set $l_j = \{(0,t); 2^{-j} \le t < 2^{-j+1}\}$. Then $C_{G_\alpha}(l_j) = 2^{-j(2-\alpha)} C_{G_\alpha}(l_0)$ on account of Lemma 1. Hence it suffices to show that $C_{G_\alpha}(l_0) > 0$ in case $1 < \alpha \le 2$.

Suppose $1 < \alpha \le 2$. We first note that $u(y) = \int_0^2 G_\alpha((0,x_2), y) \, dx_2$ is a bounded and continuous function of $y \in D$, so that $u(y) \le M$ for any $y \in D$ with a positive constant M. If μ is a nonnegative measure on D such that $G_\alpha \mu(x) \ge 1$ for every $x \in l_0$, then

$$\mu(D) \ge M^{-1} \int_D u(y) \, d\mu(y) \ge M^{-1} \int_0^2 G_\alpha \mu((0,x_2)) \, dx_2 \ge M^{-1}.$$

Hence it follows that $C_{G_\alpha}(l_0) \ge M^{-1} > 0$.

LEMMA 4. *Let γ be a bounded curve in the upper half-plane D. If the closure of γ in the plane has a point on ∂D, then γ is not α-thin at ∂D when $1 < \alpha \leq 2$.*

This lemma is an easy consequence of Lemmas 2 and 3. Next we are concerned with the best possibility of our theorem as to the size of the exceptional sets.

PROPOSITION. *Let E be a bounded set in D. If E is α-thin at ∂D, then there exists a nonnegative measure μ on D satisfying* (1) *and*

$$\lim_{x_n \to 0, x \in E} x_n^{n-\alpha+1} G_\alpha \mu(x) = \infty.$$

PROOF. By the definition of C_{G_α}, for each positive integer j we can find a nonnegative measure μ_j on D such that $\mu_j(D) < C_{G_\alpha}(E_j) + \varepsilon_j$ and $G_\alpha \mu_j(x) \geq 1$ for any $x \in E_j$, where $\{\varepsilon_j\}$ is a sequence of positive numbers such that

$$\sum_{j=1}^{\infty} 2^{j(n-\alpha)} \{C_{G_\alpha}(E_j) + \varepsilon_j\} < \infty.$$

Further we can find a sequence $\{a_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j = \infty$ and $\sum_{j=1}^{\infty} a_j 2^{j(n-\alpha)} \{C_{G_\alpha}(E_j) + \varepsilon_j\} < \infty$. Letting $R > 1$ be a positive number such that $E \subset B(0, R)$, we denote by μ_j' the restriction of μ_j to the set $\{y \in B(0, 2R); 2^{-j-1} < y_n < 2^{-j+2}\}$ and define $\mu_j'' = \mu_j - \mu_j'$. If $x \in E_j$, then $G_\alpha \mu_j''(x) \leq$ const. $2^{j(n-\alpha)} \mu_j''(D)$, so that

$$G_\alpha \mu_j''(x) < 1/2 \quad \text{for } j \text{ sufficiently large.}$$

Set $\mu = \sum_{j=1}^{\infty} a_j 2^{j(n-\alpha+1)} \mu_j'$. Then it follows that $\int_D y_n \, d\mu(y) < \infty$, so that μ satisfies (1). Moreover, if $x \in E_j$, then we have

$$x_n^{n-\alpha+1} G_\alpha \mu(x) \geq 2^{-j(n-\alpha+1)} G_\alpha \mu(x) \geq a_j G_\alpha \mu_j'(x) \geq a_j/2,$$

which implies that $\lim_{x_n \to 0, x \in E} x_n^{n-\alpha+1} G_\alpha \mu(x) = \infty$. Thus μ satisfies all the conditions in the proposition.

REMARK 1. For a set $E \subset D$, E is n-thin at ∂D if and only if $C_{G_n}(E) < \infty$. Thus our theorem together with our proposition implies the following result (cf. Luecking [**3**]): *Let $F \subset D$ be bounded. Then $C_{G_n}(F) = \infty$ if and only if $\lim_{x_n \to 0, x \in F} x_n G_n \mu(x) = 0$ for any nonnegative measure μ on D such that $G_n \mu \not\equiv \infty$.*

To prove that $C_{G_n}(E) < \infty$ implies the n-thinness of E at ∂D, let μ be a nonnegative measure on D such that $\mu(D) < \infty$ and $G_n \mu(x) \geq 1$ for any $x \in E$. As in the proof of the Theorem, write $G_n \mu = u_1 + u_2$. Since $\mu(D) < \infty$ and $G_n(x, y)$ remains bounded when $|x - y| \geq x_n/2$, we see, as in the proof of the Theorem, that $\lim_{x_n \to 0} u_2(x) = 0$. Hence, if j is sufficiently large, then $u_1(x) > 1/2$ for $x \in E_j$. This implies that

$$C_{G_n}(E_j) \leq 2 \int_{\{y \in D; 2^{-j-1} < y_n < 2^{-j+2}\}} d\mu(y).$$

Thus it follows that E is n-thin at ∂D. Since C_{G_n} is countably subadditive, it is clear that the n-thinness of E at ∂D implies $C_{G_n}(E) < \infty$.

REMARK 2. In the upper half-plane we consider the Green potential $u(x) = \int_0^\infty G_\alpha(x,(0,t))t^{-\beta}\,dt$. If $\alpha - 2 < \beta < 2$, then

$$\lim_{r\downarrow 0} r^{2-\alpha+(\beta-1)}u(0,r) = u(0,1) < \infty.$$

This example will show the best possibility of our Theorem as to the order of infinity.

REMARK 3. In view of the proof of the Theorem, we can prove that for a nonnegative measure μ on D satisfying (1), there exists a set $E \subset D$ which is α-thin at ∂D and satisfies

$$\lim_{x_n\to 0, x\in D-E} x_n^{n-\alpha+1}(1+|x|)^{\alpha-n-2}G_\alpha\mu(x) = 0.$$

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