ORTHOGONAL COMPLEX STRUCTURES ON $S^6$

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ABSTRACT. A complex structure on the six-sphere is called orthogonal if the standard metric is Hermitian with respect to it. While such structures locally exist in profusion, there is no such complex structure on the entire sphere.

An almost-complex structure $J$ on a smooth manifold $M$ (of necessarily even dimension) is an endomorphism $J : TM \rightarrow TM$ of the tangent bundle such that $J^2 = -1$; such a structure identifies $TM$ with a complex vector bundle by letting scalar multiplication by $\sqrt{-1}$ be defined to be the endomorphism $J$. It is known [1] that the only spheres admitting such structures are $S^2$ and $S^6$; such structures are naturally defined if one takes as models of $S^2$ and $S^6$, respectively, the sets of quaternions and octonians with square $-1$, allowing one to define the tensor $J$ at $x$ simply to be multiplication by $x$.

The above structure on $S^6$ is not integrable in the sense that one cannot find local charts in which $J$ becomes multiplication by $\sqrt{-1}$ in $\mathbb{C}^3$. In fact it would be a minor disaster if $S^6$ were to admit the structure of a complex 3-manifold in the following sense: If we then blew up a point we would obtain a complex 3-manifold diffeomorphic to $\mathbb{CP}^3$ which would not be biholomorphically equivalent to $\mathbb{CP}^3$. (Notice that we can carry out this process on the level of almost-complex structures, thereby obtaining an “exotic” almost-complex structure on $\mathbb{CP}^3$ with $c_1^3 = -8$, in contrast to the usual structure with $c_1^3 = 64$.) This would stand in marked contrast to the result [2] that $\mathbb{CP}^3$ has only one isomorphism class of Kähler complex structures.

In this note, we restrict our attention to a subclass of almost-complex structures on the six-sphere, namely those which act on the tangent space as isometries with respect to the usual “round” metric; such an almost complex structure will be called orthogonal. Thus, the standard example cited above is an orthogonal almost-complex structure. We will show that no globally defined orthogonal almost-complex structure is integrable.

Let $J$ be any almost-complex structure on $U \subset S^6$, and consider the complex vector bundle $T^{0,1} \rightarrow U$ defined as the $-i$ eigenspace of $J$:

$$T^{0,1} = \{v \in (TS^6) \otimes \mathbb{C} | Jv = -iv\}.$$

Since $T_xS^6 \subset \mathbb{R}^7$ for every $x \in S^6$, $T_x^{0,1} \subset \mathbb{C}^7$, and we have a tautological map

$$\tau : U \rightarrow G_3(\mathbb{C}^7)$$

into the Grassmannian of 3-planes in $\mathbb{C}^7$. If $\mathcal{G} \subset G_3(\mathbb{C}^7)$ is the open subset consisting of planes $P$ for which $P \cap \overline{P} = \{0\}$, $\tau(S^6) \subset \mathcal{G}$. But if we define a
projection $\pi: S^6 \to S^6$ by assigning to a complex 3-plane $P$ the oriented normal of its real projection $(P + \bar{P}) \cap \mathbb{R}^7$, it follows that $\pi \tau: U \to U$ is the identity, so $\tau$ is necessarily an embedding.

**Lemma.** Let $J$ be orthogonal with respect to the usual metric on $S^6 \subset \mathbb{R}^7$. If $J$ is integrable then $\tau$ is a holomorphic map.

**Proof.** If $J$ is orthogonal and $V$ and $W$ are two elements of $T^0_1$, one has

$$g(V, W) = g(JV, JW) = g(-iV, -iW) = -g(V, W)$$

so that $g(V, W) = 0$, where $g$ is the usual metric on $\mathbb{R}^7$ extended to $\mathbb{C}^7$ by complex linearity in both factors. Let $z^\alpha$, $\alpha = 1, 2, 3$, be local holomorphic coordinates on some region of $S^6$ and let $X_\alpha$ be complex vector fields on $S^6$ determined by the equations

$$X_\alpha \nabla a(X_\beta)_b = X_\alpha \nabla a(dz^\alpha)_b = X_\alpha \nabla a(dz^\alpha)_a$$

so that $\nabla X_\alpha X_\beta = -\nabla X_\beta X_\alpha$, and

$$[X_\alpha, X_\beta] = \nabla X_\alpha X_\beta - \nabla X_\beta X_\alpha = 2\nabla X_\alpha X_\beta.$$

But the integrability condition implies that $[X_\alpha, X_\beta]$ is a section of $T^0_1$. Hence

$$\nabla X_\alpha X_\beta \equiv 0 \mod \text{span}\{X_\beta\}.$$ 

But the Levi-Civita connection $\nabla$ on the unit sphere is related to the flat connection $D$ on $\mathbb{R}^7$ by $\nabla_W V = D_W V + g(V, W)N$, where $N$ is the unit outward-pointing normal vector field of $S^6$; hence

$$D_X_\alpha X_\beta = \nabla X_\alpha X_\beta - g(X_\alpha, X_\beta)N = \nabla X_\alpha X_\beta$$

$$= \frac{1}{2}[X_\alpha, X_\beta] \equiv 0 \mod \text{span}\{X_\gamma\}.$$ 

Since $\text{Hom}(T^0_1, \mathbb{C}^7/T^0_1)$ is precisely the holomorphic tangent space of $G_3(\mathbb{C}^7)$ at $T^0_1$ the above calculation shows that all the $(0, 1)$-components of the derivative of $\tau$ vanish, and $\tau$ is holomorphic. Q.E.D.

As corollary we get the following result:

**Theorem.** $S^6$ has no integrable orthogonal complex structure.

**Proof.** Suppose it did. Then $\tau$ would be an embedding of $S^6$ in $G_3(\mathbb{C}^7)$ as a complex manifold. But Grassmannians are Kähler manifolds, so this would give $S^6$ a Kähler structure. But this is impossible because $H^2(S^6) = 0$. Q.E.D.

**Remark.** For the lemma the dimension of $S^6$ was inessential; if we applied the same technique to the usual orthogonal complex structure on $S^2$ we would obtain a holomorphic map $\tau: S^2 \to G_2(\mathbb{C}^3) = \mathbb{CP}_2$, thereby realizing the 2-sphere as a
nonsingular plane conic. Of course, the contradiction of the theorem does not occur in this case because $H^2(S^2) \neq 0$.

It remains to be seen whether techniques of this kind may be generalized to the nonorthogonal case.

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REFERENCES


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