ABSTRACT. We prove some results concerning the covering, additivity, and
the uniform numbers for general topological spaces.

For any dense-in-itself $T_1$ topological space $X$ let us put $\text{cov}(X) = \inf\{|\mathcal{R}|: \mathcal{R}

\text{is a family of nowhere dense subsets of } X \text{ covering } X\}$. Then $\text{cov}(X)$ is always an
infinite cardinal that does not exceed $|X|$ , the cardinality of $X$. If $\text{cov}(X)$ is an
uncountable cardinal, then $X$ is second category, and if $\text{cov}(U)$ is uncountable for
every nonempty open subset of $X$, then $X$ is a Baire space.

Let us put $\text{cov}^2(X) = \inf\{\text{cov}(F): F \text{ is a closed, nonempty, dense-in-itself sub-

space of } X\}$. If $\text{cov}^2(X)$ is an uncountable cardinal, then $X$ is usually called a totally
nonmeager space [1] (sometimes it is also called a totally inexhaustible space [8]).

In [8], the behavior of totally nonmeager spaces under feebly continuous and
feebly open mappings have been considered (see also [2]). Let us recall [4] that
a mapping $f$ of a space $X$ onto a space $Y$ is feebly continuous if $\text{int } f^{-1}(V) \neq \emptyset$
whenever $V$ is nonempty and open in $Y$, and $f$ is feebly open if $\text{int } f(U) \neq \emptyset$.
whenever $U$ is nonempty and open in $X$. T. Neubrunn [9] conjectured that

(*) If $f$ is a one-to-one feebly continuous and feebly open mapping of a regular
space $X$ onto a totally nonmeager space $Y$, then $X$ is a Baire space.

We shall show that this conjecture is true even without appealing to any addi-
tional separation axioms. For this purpose we prove the following general result.

**THEOREM 1.** Let $f$ be a one-to-one feebly continuous and feebly open mapping
of a space $X$ onto a space $Y$. If $U$ is a nonempty open subset of $X$, then $\text{cov}^2(Y) \leq
\text{cov}(U)$.

**PROOF.** Suppose that $U$ is a nonempty open subset of $X$, $\mathcal{R}$ is a family of
nowhere dense subsets of $U$ that covers $U$, and $|\mathcal{R}| = \text{cov}(U)$. Let us put $W = \text{int cl int } f(X - \text{cl } U)$. Then $W$ is an open subset of $Y$ that is disjoint with $\text{int } f(U)$, $f$
being one-to-one. We shall show that $\text{cov}(F) \leq \text{cov}(U)$, where $F = Y - W$.
The closed set $F$ can be decomposed into two disjoint parts: $F_1 = \text{int } f(U)$ and
$F_2 = F - \text{int } f(U)$. Part $F_2$ is a closed subset of the closed set $F$. Let us check that
it is also nowhere dense in $F$. So let $V$ be an open subset of $Y$ such that $F_2 \cap V \neq \emptyset$.
Hence $V - W \neq \emptyset$ and, because $V$ is open $G = V - \text{cl int } f(X - \text{cl } U) \neq \emptyset$.
Because $f$ is feebly continuous, $\text{int } f^{-1}(G) \neq 0$. This set has to be disjoint with $X - \text{cl } U$;
otherwise $G \cap f(X - \text{cl } U) \neq \emptyset$ ($f$ being feebly open), which is impossible. Hence
$\text{int } f^{-1}(G) \subset \text{cl } U$, and therefore $\text{int } f^{-1}(G) \cap U \neq \emptyset$. Because $f$ is feebly open,
G \cap \text{int } f(U) \neq \emptyset. Since G \subset V, V \cap \text{int } f(U) \neq \emptyset which shows that \( F_2 \) is nowhere dense in \( F \).

To prove our theorem now, it is enough to show that int \( f(U) \) can be covered by cov(\( U \)) or a less nowhere dense subset of \( F \). We will have this if we show that for any \( E \in \mathcal{R} \), \( F_E = \text{int } f(U) \cap f(E) \) is a nowhere dense subset of \( F \). To see this we argue in the following way. The image of a nowhere dense subset of \( X \) under a one-to-one feebly open and feebly continuous mapping is again a nowhere dense subset of \( Y \). Nowhere dense subsets of an open set of a space are exactly those sets which are subsets of the open set and nowhere dense in the space. Therefore each \( F_E \) is a nowhere dense subset of int \( f(U) \), being the intersection of a nowhere dense set in the space \( Y \) with an open subset of the space \( Y \). Because int \( f(U) \) is contained in \( F \), those sets are also nowhere dense in \( F \). \( \square \)

As we know, cov(\( Y \)) > \( \omega \) means the same as \( Y \) being totally nonmeager, and cov(\( U \)) > \( \omega \) for each nonempty open subset of \( X \) means the same as \( X \) being a Baire space, so (*) follows immediately from Theorem 1.

Our next topic is to consider the cardinal \( \text{add}(X) \), the additivity of the category for the space \( X \). In order for \( \text{add}(X) \) to be defined we restrict ourselves to the case in which cov(\( X \)) is defined and cov(\( X \)) > \( \omega \). In such a case we put

\[
\text{add}(X) = \inf \left\{ |\mathcal{R}| : \mathcal{R} \text{ is a family of nowhere dense subsets of } X \right. \quad \text{and} \left. \bigcup \mathcal{R} \text{ is not first category in } X \right\}.
\]

There is an obvious inequality: \( \text{add}(X) \leq \text{cov}(X) \). Both cardinals \( \text{add}(X) \) and cov(\( X \)) have been intensively studied for \( X \) being a metric separable space (see, for example, [6, 7]). Many interesting and deep results have been obtained. One of the most celebrated results in this topic is the following inequality:

(***). If \( \kappa \) is a cardinal such that \( P(\kappa) \) holds and \( X \) is a separable, metric second category space, then \( \text{add}(X) > \kappa \).

Here \( P(\kappa) \) denotes the following combinatorial statement:

\( P(\kappa) \): If \( S \) is a family of infinite subsets of \( \omega \) such that \( |S| \leq \kappa \), and any finite subfamily of \( S \) has an infinite intersection, then there is an infinite subset \( A \) of \( \omega \) such that \( A - s \) is finite for each \( s \in S \).

The result (*** has been shown by D. Martin and R. Solovay [5]. We shall present a theorem that generalizes (***). Our generalization goes in two directions. First, it concerns a much wider class of spaces. Second, it uses a weaker assumption than \( P(\kappa) \)—namely, we shall use the following statement:

\( P(\kappa) \downarrow \). If \( \{T_\alpha : \alpha < \lambda \} \) is a family of infinite subsets of \( \omega \) such that \( \lambda \leq \kappa \) and \( \alpha < \beta < \lambda \) implies \( T_\beta - T_\alpha \) is finite, then there is an infinite subset \( A \) of \( \omega \) such that \( A - T_\alpha \) is finite for each \( \alpha < \lambda \).

Clearly, \( P(\kappa) \downarrow \) follows from \( P(\kappa) \). It is known [3] that \( P(\omega_1) \) follows from \( P(\omega_1) \downarrow \). However it is unknown whether \( P(\kappa) \) is equivalent to \( P(\kappa) \downarrow \), in general.

**Theorem 2.** If \( P(\kappa) \downarrow \) holds and \( X \) is a second category space containing a dense countable subspace of points with countable character, then \( \text{add}(X) > \kappa \).

To prove this theorem we will need two lemmas. The first of these is already known; it goes back to F. Rothberger [10] though the explicit proof can be found in [3].
LEMMA 1. Let $\mathcal{F}$ be a family of functions from $\omega$ into $\omega$ such that $|\mathcal{F}| \leq \kappa$. If $P(k) \downarrow$ holds, then there is a function $g$ from $\omega$ into $\omega$ such that the set $\{n \in \omega : g(n) \leq f(n)\}$ is finite for each $f \in \mathcal{F}$. □

LEMMA 2. Let $X$ be a dense-in-itself $T_1$ space with a countable $\pi$-base, and let $\{D_\alpha : \alpha < \lambda\}$ be a family of countable dense subsets of $X$ such that $D_\beta - D_\alpha$ is finite whenever $\alpha < \beta < \lambda$. If $\lambda \leq \kappa$ and $P(\kappa) \downarrow$ holds, then there is dense subset $D$ of $X$ such that $D - D_\alpha$ is finite for each $\alpha < \lambda$.

PROOF. Let $\mathcal{P}$ be a countable $\pi$-base in $X$ consisting of nonempty open sets. For each $U \in \mathcal{P}$ let us consider the family $\{U \cap D_0 \cap D_\alpha : \alpha < \lambda\}$. Each such family consists of infinite subsets of the countable set $U \cap D_0$. This holds because $X$ is $T_1$ and the sets $D_\alpha$ are dense in $X$. We also have $U \cap D_0 \cap D_\beta = U \cap D_0 \cap D_\alpha$ is finite, whenever $\alpha < \beta < \lambda$. So we may apply $P(\kappa) \downarrow$, and we get that for any $U \in \mathcal{P}$ there is an infinite subset $D_U$ of $U \cap D_0$ such that $D_U - D_\alpha$ is finite for each $\alpha < \lambda$. Enumerate $D_U$ by $\{d(U, n) : n \in \omega\}$. Now for each $\alpha < \lambda$ define a function $f_\alpha : \mathcal{P} \to \omega$, setting $f_\alpha(U) = \min\{n : d(U, m) \in D_\alpha \text{ for each } m \geq n\}$. Since $D_U - D_\alpha$ is finite for each $U \in \mathcal{P}$, the function $f_\alpha$ is well defined. We may apply our Lemma 1 to the family $\{f_\alpha : \alpha < \lambda\}$, and we get the existence of a function $g$ from $\mathcal{P}$ into $\omega$ such that the set $\{U \in \mathcal{P} : g(U) \leq f_\alpha(U)\}$ is finite for each $\alpha < \lambda$. Let us put $D = \{d(U, n) : U \in \mathcal{P} \text{ and } g(U) \leq n\}$. Then $D$ intersects each $D_U$ on an infinite set and therefore $D$ intersects each $U \in \mathcal{P}$ on an infinite set. Since $\mathcal{P}$ is a $\pi$-base in $X$, $D$ is dense in $X$. It remains to be shown that $D - D_\alpha$ is finite for each $\alpha < \lambda$. So fix $\alpha$, $\alpha < \lambda$. Let us observe that if $d(U, n) \in D - D_\alpha$, then $g(U) \leq f_\alpha(U)$ and $d(U, n) \in D - D_\alpha$. Hence $D - D_\alpha \subset \bigcup\{D_U - D_\alpha : U \in \mathcal{P} \text{ and } g(U) \leq f_\alpha(U)\}$. Because the sets $D_U - D_\alpha$ are finite and the set $\{U \in \mathcal{P} : g(U) \leq f_\alpha(U)\}$ is finite as well, $D - D_\alpha$ is a finite set.

PROOF OF THEOREM 2. Assume, to the contrary, that there exists in the space $X$ a family $\mathcal{R}$ consisting of nowhere dense subsets of $X$ such that $\bigcup \mathcal{R}$ is second category in $X$ and yet $|\mathcal{R}| \leq \kappa$. Without loss of generality we may assume that $X$ is dense-in-itself (otherwise apply the arguments below to the set $\text{int} \bigcup \mathcal{R}$). Let $D$ be a countable dense subset of $X$ consisting of points with countable character. For each $d \in D$ enumerate a countable base around $d$ by $\{U(d, n) : n \in \omega\}$. We may also assume that $U(d, n) \subset U(d, m)$, whenever $m \leq n$. Of course, the family $\{U(d, n) : d \in D \text{ and } n \in \omega\}$ forms a countable $\pi$-base in $X$. Knowing this we will be able to find a dense subset $C$ contained in $D$ such that $C \cap \text{cl} F$ is finite for each $F \in \mathcal{R}$. For this purpose enumerate the family $\mathcal{R}$, say $\mathcal{R} = \{F_\alpha : \alpha < \lambda\}$. Since $|\mathcal{R}| \leq \kappa$, $\lambda \leq \kappa$. And now, we shall inductively define a family $\{D_\alpha : \alpha < \lambda\}$ satisfying the following conditions:

(i) $D_\alpha \subset D$, $D_\alpha$ is dense in $X$, and $D_\alpha \cap \text{cl} F_\alpha = \emptyset$ for each $\alpha < \lambda$;
(ii) if $\alpha < \beta < \lambda$, then $D_\beta - D_\alpha$ is finite.

Put $D_0 = D - \text{cl} F_0$. Assume $\gamma < \lambda$ and $\{D_\alpha : \alpha < \gamma\}$ has already been defined. If $\gamma$ is a nonlimit ordinal, say $\gamma = \beta + 1$, then it is enough to put $D_\gamma = D_\beta - \text{cl} F_\gamma$. If $\gamma$ is a limit ordinal, then let $\{\alpha_\xi : \xi < \text{cf}(\gamma)\}$ be a strongly increasing sequence of ordinals less than $\gamma$. Because $\text{cf}(\gamma) < \lambda \leq \kappa$ and $\text{cf}(\gamma)$ is a cardinal, we may apply our Lemma 2 to the family $\{D_{\alpha_\xi} : \xi < \text{cf}(\gamma)\}$. Hence we get a dense subset $D_\gamma$ of $X$ such that $D_\gamma - D_\alpha$ is finite for each $\xi < \text{cf}(\gamma)$. We shall show that $D_\gamma - D_\alpha$ is finite for each $\alpha < \gamma$. So fix $\alpha$, $\alpha < \gamma$. There is $\xi$, $\xi < \text{cf}(\gamma)$, such that $\alpha < \alpha_\xi$.
Hence
\[ \tilde{D}_\gamma - D_\alpha \subset \tilde{D}_\gamma - (D_\alpha \cap D_{\alpha'\epsilon}) \subset (\tilde{D}_\gamma - D_{\alpha'\epsilon}) \cup (D_{\alpha'\epsilon} - D_\alpha) \]
which is a finite set. Finally, if we put \( D_\gamma = (\tilde{D}_\gamma \cap D) - \operatorname{cl} F_\gamma \) we get the desired set to complete the induction step.

Now, if we have a family \( \{D_\alpha: \alpha < \lambda\} \) satisfying conditions (i) and (ii), we shall again apply our Lemma 2 to this family (since \( \lambda \leq \kappa \)), and we get a dense subset \( C \) of \( X \) such that \( C - D_\alpha \) is finite for each \( \alpha < \lambda \). We may assume that \( C \) is a subset of \( D \) (throwing out finitely many points if needed); the properties \( C - D_\alpha \) is finite and \( D_\alpha \cap \cl F_\alpha = \emptyset \) give us that \( C \cap \cl F_\alpha \) is finite, so \( C \cap \cl F \) is finite for each \( F \in \mathcal{R} \).

Enumerate the points of \( C \) by \( \{c_n: n \in \omega\} \) and consider the families \( \mathcal{R}_n = \{F \in \mathcal{R}: C \cap \cl F \subset \{c_0, \ldots, c_n\}\} \). Each member of \( \mathcal{R} \) falls into one of the families \( \mathcal{R}_n \). Hence one of them, say \( \mathcal{R}_m \), has the union of its members being second category. Let us throw out \( m+1 \) first points from the set \( C \). Then we obtain a dense countable set \( B \) such that \( B \cap \cl F = \emptyset \) for each \( F \in \mathcal{R}_m \). Now for each \( F \in \mathcal{R}_m \) define a function \( f_F: B \to \omega \) setting \( f_F(b) = \min\{n: U(b, n) \cap \cl F = \emptyset\} \) (recall that \( B \) is a subset of \( D \) and for each \( d \in D \) we have assigned a local base, \( \{U(d, n): n \in \omega\} \)).

Since \( b \notin \cl F \) for each \( F \in \mathcal{R}_m \), the function \( f_F \) is well defined. We may apply our Lemma 1 to the family \( \{f_F: F \in \mathcal{R}_m\} \), and we get the existence of a function \( g: B \to \omega \) such that the set \( \{b \in B: f(b) \leq f_F(b)\} \) is finite for each \( F \in \mathcal{R}_m \). For each finite subset \( S \) of the set \( B \) let us put \( E_S = X - \bigcup\{U(b, g(b)): b \in B - S\} \). The number of the sets \( E_S \) is countable, since \( B \) is a countable set. So we will come to a contradiction if we show that \( \bigcup \mathcal{R}_m \subset \bigcup \{E_S: \text{\( S \) is a finite subset of \( B \)}\} \) and that each set \( E_S \) is nowhere dense in \( X \).

To see that \( \bigcup \mathcal{R}_m \subset \bigcup \{E_S: \text{\( S \) is a finite subset of \( B \)}\} \), let us observe that \( F \subset E_S \), whenever \( S = \{b \in B: g(b) \leq f_F(b)\} \).

To see that \( E_S \) is nowhere dense, take an arbitrary nonempty open subset \( V \) of \( X \). The \( V \cap B \) is infinite and therefore there is a point \( b_0 \) belonging to \( V \cap B - S \). Consequently,
\[ b_0 \in V \cap U(b_0, g(b_0)) \subset V \cap \bigcup\{U(b, g(b)): b \in B - S\}. \]

Our final topic is to consider the cardinal \( u(X) \). It is defined (if possible) in the following way:
\[ u(X) = \inf\{|Y|: Y \text{ is a second category subset of } X\}. \]

There are obvious inequalities between \( \text{add}(X) \) and both cardinals \( \text{cov}(X) \) and \( u(X) \) in the case in which \( \text{cov}(X) \) is defined, and \( X > \omega \), namely \( \text{add}(X) \leq u(X) \) and \( \text{add}(X) \leq \text{cov}(X) \). Within ZFC one cannot prove the equalities \( \text{add}(X) = u(X) \) or \( u(X) = \text{cov}(X) \), even if \( X \) is the space of reals [6].

\( M(\kappa) \). If \( \mathcal{F} \) is a family of functions from \( \omega \) into \( \omega \) such that \( |\mathcal{F}| \leq \kappa \), then there is a function \( g \) from \( \omega \) into \( \omega \) such that the set \( \{n \in \omega: g(n) = f(n)\} \) is finite for each \( f \in \mathcal{F} \).

**Theorem 3.** If \( M(\kappa) \) holds and \( X \) is a dense-in-itself second category Hausdorff space with a countable \( \pi \)-base, then \( u(X) > \kappa \).

**Proof.** Assume, to the contrary, that the space \( X \) contains a second category subset \( Y \) such that \( |Y| \leq \kappa \). Let \( \mathcal{P} \) be a countable \( \pi \)-base in \( X \). Because \( X \) is a
dense-in-itself Hausdorff space, each nonempty open subset of $X$ contains infinitely many disjoint nonempty open subsets. For any $u \in \mathcal{P}$, $u \neq \emptyset$, let $\{u(n): n \in \omega\}$ be a disjoint family of nonempty open subsets of the set $u$. Now, for any $y \in Y$ define a function $f_y: \mathcal{P} - \{\emptyset\} \to \omega$ setting $f_y(u) = n$ if $y \in u(n)$ and $f_y(u) = 0$ if $y \notin \bigcup\{u(n): n \in \omega\}$. Because $|Y| \leq \kappa$ and $M(\kappa)$ holds, there is a function $g$ from $\mathcal{P} - \{\emptyset\}$ into $\omega$ such that the set $\{u \in \mathcal{P} - \{\emptyset\}: g(u) = f_y(u)\}$ is finite for each $y \in Y$. For each finite subset $s$ of the set $\mathcal{P}$ let us put $E_s = X - \bigcup\{u(g(u)): u \in \mathcal{P} - s\}$. The number of the sets $E_s$ is countable. So, we will come to a contradiction if we show that $Y \subseteq \bigcup\{E_s: s$ is a finite subset of $\mathcal{P}\}$ and that each set $E_s$ is nowhere dense. To see the first observe that if $y \in Y$, then $y \in E_s$, where $s = \{u \in \mathcal{P} - \{\emptyset\}: g(u) = f_y(u)\}$. To see that each $E_s$ is nowhere dense, fix $s$ and take an arbitrary nonempty open subset $V$ of $X$. Because $X$ is dense in itself, the set of all members of $\mathcal{P}$ contained in $V$ is infinite, and therefore there is one in $\mathcal{P} - (s \cup \{\emptyset\})$. Hence $V \cap \bigcup\{u(g(u)): u \in \mathcal{P} - s\} \neq \emptyset$.

In [6, Theorem 1.3] some analogous results have been obtained for metric separable spaces.

**References**