AN ADJOINT REPRESENTATION
FOR POLYNOMIAL ALGEBRAS

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ABSTRACT. This paper shows that a graded polynomial algebra over $F_2$ with
Steenrod algebra action possesses an analog of the adjoint representation for
the cohomology of the classifying space of a compact connected Lie group.

1. Introduction. Let $R = F_2[z_1, z_2, \ldots, z_n]$ be a graded polynomial algebra
over the field with two elements and, in addition, suppose that $R$ has an action
of the mod 2 Steenrod algebra $A$ for which it is an unstable algebra over $A$. This
condition means that $R$ has the formal algebraic properties satisfied by the mod 2
cohomology of a space. Specifically, there are homomorphisms $Sq^i : R^j \to R^{j+i}$ for
$i \geq 0$ satisfying

\[ (1) \quad Sq^i(xy) = \sum_{j=0}^{i} Sq^j x \cdot Sq^{i-j} y, \]

and

\[ (2) \quad Sq^i x = \begin{cases} 
  x & \text{if } i = 0, \\
  x^2 & \text{if } i = \dim x, \\
  0 & \text{if } i > \dim x.
\end{cases} \]

It is convenient to let $Sq = 1 + Sq^1 + Sq^2 + \cdots$ be the total Steenrod square, which
is then an algebra automorphism of $R$. As examples we have:

(1) Let $R_0 = F_2[x_1, \ldots, x_n]$ with $\dim x_i = 1$. The action of $A$ is then uniquely
described by $Sq x_i = x_i + x_i^2$, and $R_0$ is isomorphic to $H^*(BZ_2; Z_2)$, the mod 2
cohomology of the classifying space for the group $Z_2^n$.

(2) Let $R_1 = (F_2[x_1, \ldots, x_n])^{\Sigma_n}$ be the ring of invariants of the symmetric group
acting to permute the $x_i$. Then $R_1 = F_2[w_1, \ldots, w_n]$, where $w_i$ is the $i$th elementary
symmetric function of the $x_j$. Then $R_1$ is isomorphic to $H^*(BO(n); Z_2)$, where $O(n)$
is the orthogonal group, with $w_i$ being called the $i$th universal Stiefel-Whitney class.

(3) Let $R_2 = D_n = (F_2[x_1, \ldots, x_n])^{GL_n} F_2$ be the ring of invariants of the general
linear group $GL_n F_2$, considered as the linear transformations of the span of
$x_1, \ldots, x_n$. Then $D_n = F_2[c_{n'0}, c_{n'1}, \ldots, c_{n'n}]$, where $\dim c_{n'i} = 2^n - 2^i$, is called the
$n$th Dickson algebra. Except for small values of $n$ it is not the mod 2 cohomology
of a space.

The prototypical example is, of course, given by $R = H^*(BG; Z_2)$, which is the
mod 2 cohomology of the classifying space of a compact Lie group. For general $G$
the cohomology is not polynomial, but we are considering the polynomial situation.
In this situation, with $R = F_2[z_1, \ldots, z_n]$, $n$ is the mod 2 rank of $G$ (see \cite{Q}) and

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When $G$ is a compact Lie group, we have the adjoint representation giving a homomorphism $\text{Ad}: G \rightarrow O(m)$. This induces a map $B\text{Ad}: BG \rightarrow BO(m)$, with an induced homomorphism $H^*(BO(m);Z_2) \rightarrow H^*(BG;Z_2)$ or equivalently a homomorphism

$$\text{Ad}: F_2[w_1, w_2, \ldots, w_m] \rightarrow R.$$ 

The objective of this paper is to show that such a homomorphism actually arises directly from the algebraic structure of $R$. In particular, a polynomial algebra which is an unstable algebra over the mod 2 Steenrod algebra has a homomorphism

$$\text{Ad}: F_2[w_1, \ldots, w_m] \rightarrow R = F_2[z_1, \ldots, z_n]$$

which is an analogue of the adjoint representation.

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2. A definition. If $G$ is a compact Lie group, we have the adjoint representation $\text{Ad}: G \rightarrow O(m)$ and hence an induced homomorphism $\text{Ad}: H^*BO(m) \rightarrow H^*BG$, where we let $H^*BG = H^*(BG;Z_2)$. Of course, one way to obtain this homomorphism is to consider the representation $\text{Ad}$ as giving a vector bundle $EG \times_G R^m \rightarrow EG/G = BG$. This vector bundle has a Thom space $T\text{Ad}_G$, and the reduced mod 2 cohomology of that Thom space $H^*T\text{Ad}_G$ is a free $H^*BG$ module of rank one, with generator a Thom class $U \in H^{2m}T\text{Ad}_G$. Then $Sq^*U = (1 + w_1 + \cdots + w_m)U,$ where the class $w_i$ is obtained by writing $Sq^*U$ as a multiple of $U$.

More generally, a homomorphism $H^*BO(m) \rightarrow R$ always arises from a Thom module over $R$, i.e. a module over the semitensor product $r \otimes A$ that is free of rank one as an $r$-module (Handel [H]). We are then asking for a Thom module for $R$ analogous to the adjoint Thom module $H^*T\text{Ad}_G$ for $H^*BG$.

**THEOREM.** Let $R$ be a polynomial algebra of rank $n$ and dimension $m$ over the mod 2 Steenrod algebra. Then there is a Thom module $\text{Ad}_R$ over $R$ with the following properties:

(a) If $R = H^*BG$, where $G$ is either a compact connected Lie group or $O(n)$, then $\text{Ad}_R \cong H^*T\text{Ad}_G$.

(b) $\dim \text{Ad}_R \leq m$; i.e. $w_i = 0$ if $i > m$.

(c) If $Q$ is another polynomial $A$-algebra, $\text{Ad}_Q \otimes R = \text{Ad}_Q \otimes \text{Ad}_R$.

(d) $\text{Ad}_R$ is trivial (i.e. $w_i = 0$ for $i > 0$) if and only if $R$ is abelian.

(e) If $T$ is abelian and $R \rightarrow T$ is a nonsingular embedding with Jacobian determinant $J$, then $\text{Ad}_R \cong R \cdot J \subset T$.

**REMARKS.** The meaning of (d) and (e) will be explained in the course of the proof. The groups other than $O(n)$ to which (a) applies are $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$, $Spin(n)$ for $n \leq 9$, $G_2$, $F_4$, and products of these (Borel [B]). While no details will be given, there are mod $p$ analogues when $p$ is odd prime.

To define $\text{Ad}_R$ we make use of the Hochschild homology $H_*R$, which is defined to be $\text{Tor}^{r \otimes R}(R, R)$ (see [C-E]). By standard homological methods $H_*R$ becomes a commutative (in the graded sense) $R \otimes A$-algebra whenever $R$ is an $A$-algebra (recall that $Sq$ is an algebra automorphism). If $R$ is a polynomial of rank $n$, we define $\text{Ad}_R = H_nR$. Now $\text{Ad}_R$ is certainly a module over $R \otimes A$, but we must
explain why it is free of rank one as an \( R \)-module. In fact, when \( R \) is a polynomial, \( H_* R \) can be described very simply as follows: Let \( \Omega^1_R \) denote the module of 1-forms on \( R \)—i.e., the free \( R \)-module on symbols \( dx, x \in R \), modulo the relation: \( d(xy) = xdy + ydx \). Thus if \( z_1, \ldots, z_n \) are polynomial generators for \( R \), \( \Omega^1_R \) is a free \( R \)-module on \( dz_1, \ldots, dz_n \). Furthermore, defining \( Sq(dx) = dS qx \), \( \Omega^1_R \) becomes a module over \( R \otimes A \). Hence the algebra \( \Omega^*_R \) of differential forms becomes an \( R \otimes A \) algebra. Note: \( \Omega^k_R \) is free of rank \( \binom{n}{k} \) as an \( R \)-module.

**PROPOSITION.** \( \Omega^*_R \cong H_* R \) as \( R \otimes A \)-algebras.

**PROOF.** Define \( \phi^1 : \Omega^1_R \to H_1 R \) by mapping \( adx \) to the class of \( a \otimes x \) in the Hochschild complex (see Loday-Quillen [L-Q]). It is easy to see that this is an isomorphism of \( R \)-modules (for any commutative algebra \( R \)) and hence, in our context, an isomorphism of \( R \otimes A \)-modules. Since \( H_* R \) is strictly anticommutative with respect to the Hochschild grading (\( x \in H_k R, k \) odd, then \( x^2 = 0 \)), \( \phi^1 \) extends to a homomorphism \( \phi : \Omega^*_R \to H_* R \) of \( R \otimes A \)-algebras. On the other hand, by using a Koszul resolution to compute \( H_* R \), it is easy to see that \( \phi \) is an isomorphism. (Indeed, this is a trivial special case of a theorem of Hochschild, Kostant, and Rosenberg [H-K-R].) Thus \( H_n R = \Omega^n_R \) is a Thom module with "Thom class" \( dz_1dz_2\cdots dz_n \) (see also [B-S]).

**PROOF OF THE THEOREM.** (a) Let \( G_c \) denote \( G \) regarded as a left \( G \)-space via the conjugation action: \( g \cdot x = gxg^{-1} \). Then it is easy to see that there is a pullback diagram up to homotopy

\[
\begin{array}{ccc}
EG \times_G G_c & \longrightarrow & BG \\
\downarrow & & \downarrow \Delta \\
BG & \longrightarrow & BG \times BG
\end{array}
\]

where \( \Delta \) is the diagonal map. (Thus, \( EG \times_G G_c \) is homotopy equivalent to the free loop space on \( BG \); see [S2].) Now if \( G \) is connected, so that \( BG \times BG \) is simply connected, the Eilenberg-Moore spectral sequence associated to (1) converges to \( H^*(EG \times_G G_c) \). Its \( E_2 \)-term is precisely \( H_* R (R = H^* BG) \), and it is a spectral sequence of \( A \)-algebras. Furthermore, the spectral sequence collapses, since it is a second-quadrant cohomology spectral sequence and \( H_* R \) is generated by \( H_0 R = R \) and \( H_1 R \). Note that \( H_n R \) is therefore a quotient of \( H^*(EG \times_G G_c) \). Now \( Ad_G = EG \times_G ad_G \), where \( ad_G \) is the adjoint representation of \( G \). Identifying \( ad_G \) with a \( G \)-invariant neighborhood \( U \) of the identity in \( G_c \), we obtain a collapse map

\[
EG \times_G G_c \xrightarrow{h} EG^+ \wedge_G (U/\partial U) \cong T(Ad_G).
\]

Let \( f \) denote the composite \( H^* T(Ad_G) \xrightarrow{h^*} H^*(EG \times_G G_c) \xrightarrow{\pi} H_n R \), where \( \pi \) is the projection. Then \( f \) is a map of \( R \otimes A \)-modules, and it only remains to show that \( f \) maps Thom class to Thom class.

To see this, consider the commutative diagram

\[
\begin{array}{ccc}
H^* T(Ad_G) & \xrightarrow{h^*} & H^* EG \times_G G_c \\
\downarrow & & \downarrow \pi \\
H^* S^{\dim G} & \longrightarrow & H^* G \\
\downarrow & & \downarrow \\
\text{Tor}_n^R(F_2, F_2) = F_2 \cdot dz_1 \cdots dz_n
\end{array}
\]

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where the left-hand square arises from the diagram

\[
T(\text{Ad}_G) \xleftarrow{} EG \times_G G_c
\]

\[
S^\text{dim} G \xleftarrow{} G
\]

and the right-hand one from the map of Eilenberg-Moore spectral sequences obtained by mapping the pullback diagram

\[
\begin{array}{ccc}
G & \xrightarrow{} & EG \\
\downarrow & & \downarrow \\
\oplus & \xrightarrow{} & BG
\end{array}
\]

into the diagram (1). Here \(S^k\) is the sphere of dimension \(k\).

The spectral sequence of (3) also collapses. Furthermore, it is easy to see that the corresponding map \(H_*R \to \text{Tor}^R(F_2, F_2)\) coincides with the projection \(H_*R \to (H_*R)/\text{RH_*R}\), where \(R\) is the augmentation ideal. In particular, the map \(H_nR \to \text{Tor}_{n}^R(F_2,F_2)\) is surjective. Since the bottom row of (2) is an isomorphism, it follows that \(\pi h^*\) is an isomorphism, as desired.

The case \(G = O(n)\) is easy to check directly—see the example below.

(b) Let \(R = F_2[z_1, \ldots, z_n]\), where \(|z_i| = d_i\). Then \(\text{Sq}^{d_i}(dz_i) = d(z_i^2) = 0\). Hence \(\text{Sq}(dz_1 \cdots dz_n) = 0\) if \(k > r\).

(c) This is a general property of Hochschild homology.

Now let \(U = F_2[x_1, \ldots, x_n]\), \(\dim x_i = 1\), with its unique unstable \(A\)-algebra structure. An \(A\)-algebra \(R\) will be called abelian if \(R\) is isomorphic to a subalgebra of \(U\) of the form \(F_2[x_1^{2i_1}, \ldots, x_n^{2i_n}]\) for some nonnegative integers \(i_1, \ldots, i_n\). If \(R\) is abelian, then clearly \(\text{Ad}_R^a\) is trivial. An embedding of polynomial \(A\)-algebras

\[
R = F_2[z_1, \ldots, z_n] \hookrightarrow T = F_2[y_1, \ldots, y_n]
\]

is nonsingular if the Jacobian determinant \(J = \det[\partial z_i/\partial y_j]\) is nonzero.

(e) An embedding \(R \xrightarrow{i_n} T\) is nonsingular if and only if the induced map \(H_nR \xrightarrow{i_n} H_nT\) is nonzero, in which case \(i_n\) is an isomorphism onto \(R \cdot (Jdy_1 \cdots dy_n)\). But if \(T\) is abelian, this Thom module can be identified with \(R \cdot J \subset T\) (in particular \(R \cdot J\) is a sub-\(A\)-module of \(T\)).

Next, one knows from the work of Adams and Wilkerson \([\text{A-W}]\) that any \(R\) of rank \(n\) can be embedded in \(U\). Of course, the embedding can be singular, but on inspection, it appears that this can only happen for trivial reasons—i.e., \(R \subset T\) for some proper abelian subalgebra \(T\) of \(U\). This led us to conjecture that every \(R\) admits a nonsingular embedding in some abelian \(T\) (of the same rank, of course); this conjecture has been proven by Wilkerson \([\text{W}]\). This result is of interest in its own right, but here we simply note that part (d) of the theorem follows immediately: Choose a nonsingular embedding of \(R\) in an abelian \(T\). Then \(\text{Ad}_R^a \cong R \cdot J\) as in (e), and if \(J \neq 1\) (i.e., \(R \neq T\)) then \(A\) acts nontrivially on \(J\). This completes the proof of the Theorem.

**Examples.** (a) Let \(R = H^*BO(n) = U^{2n} = F_2[w_1, \ldots, w_n]\). Then \(J\) is easily seen to be the Vandermonde determinant \(\prod_{i<j}(x_i + x_j)\). Hence \(\text{Sq} J = (\prod_{i<j} 1 + x_i + x_j) \cdot J\) and \(w(\text{Ad}_R) = \prod_{i<j}(1 + x_i + x_j)\) (see \([\text{B-H}, \S15.4]\)
(b) Let $R = U^{GL_n} F_2$ be the Dickson algebra. Then $R = F_2[e_1, \ldots, e_n]$, where $e_i = e_{n,n-i}$, and the $A$-action is given by

$$\text{Sq} e_i = (e_i + \cdots + e_n)(1 + e_1 + \cdots + e_n) + \sum_{j>1} e_j^2.$$ 

Then $w(\text{Ad}_R) = (1 + e_1 + \cdots + e_n)^{n-1}$. This follows either directly, or by computing $J = e_i^{n-1}$. For example, if $n = 3$ then $R \cong H^*(BG_2)$, and we have $w(\text{Ad}_G) = 1 + e_1^2 + e_2 + e_3^2$. (The formula for $\text{Sq} e_i$ can be derived using the methods of [W].)

**Remark.** The most interesting case is when $R \subset U$ is already a nonsingular (i.e., separable) embedding. In that case, we see that $\dim \text{Ad}_R$ is precisely $r$.

3. **An alternative definition.** Let $R = F_2[z_1, \ldots, z_n]$ be a polynomial algebra which is an unstable algebra over $A$, and let $S = F_2[u_1, \ldots, u_n]$ be a subalgebra invariant under $A$ and having the same rank. Then $R \otimes_S F_2 = R/R^S$, which we denote $R/S$, is a Poincaré duality algebra over the Steenrod algebra of dimension $N = \dim S - \dim R = \sum \dim u_i - \sum \dim z_i$. Let $\pi: R \to R/S$ be the quotient homomorphism. (See [SI].)

Since $R/S$ is a Poincaré duality algebra, it has normal Stiefel-Whitney classes $\overline{w}_i(R/S) \in (R/S)^i$ characterized by $\overline{w}_i(R/S)x = (\text{Sq}^{-1} x)^n$ for all $x \in (R/S)^{n-i}$. These are the Stiefel-Whitney classes corresponding to the contragredient module $(R/S)^i$, called the normal Thom module.

**Proposition.** There are unique classes $\overline{w}_i \in R^i$ having the property that for every subalgebra $S \subset R$ of the same rank, $\pi \overline{w}_i = \overline{w}_i(R/S)$ is the normal Stiefel-Whitney class.

**Proof.** $R^{2^s} = F_2[z_1^{2^s}, \ldots, z_n^{2^s}]$ is a subalgebra for which $\pi: R^i \to (R/R^{2^s})^i$ is an isomorphism if $i < 2^s$, and taking $2^s > i$ determines the class $\overline{w}_i$ by its projection. To see that this is well defined, we have inclusions

$$\begin{array}{ccc}
S^{2^s} & \longrightarrow & S \\
\downarrow & & \downarrow \\
R^{2^s} & \longrightarrow & R
\end{array}$$

for any subalgebra $S$, with $R^{2^s}/S^{2^s} \rightarrow R/S^{2^s} \xrightarrow{\pi_1} R/R^{2^s}$ and $S/S^{2^s} \rightarrow R/S^{2^s} \xrightarrow{\pi_2} R/S$, being Poincaré extensions in the sense of [M, Appendix B]. Then $\overline{w}_i(R/S^{2^s})$ and the projection of $\overline{w}_i$ in $R/S^{2^s}$ have the same image under $\pi_1$ [M, Proposition B1], so coincide, since $2^s > i$; and then applying $\pi_2$ yields that $\overline{w}_i(R/S)$ is the projection of $\overline{w}_i$ into $R/S$, since it coincides with the projection of $\overline{w}_i(R/S^{2^s})$.

**Proposition.** In $R$, $\overline{w}_j = 0$ if $j > \dim R$.

**Proof.** According to Adams-Wilkerson [A-W], we have an inclusion $R \subset U = F_2[x_1, \ldots, x_n]$ with $\dim x_i = 1$, where $U$ is the algebraic closure of $R$. By [M, Theorem B3], $\bar{w}(U) = 1$. Using standard notation we let $[M]$ denote the fundamental homology class of the Poincaré algebra $M$. We may then choose a class $a \in U$ so that $\pi(a)[U/R] = 1$. Then in the sequence

$$R/R^{2^s} \xrightarrow{i} U/R^{2^s} \xrightarrow{\pi} U/R$$

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we have $r[R/R^2] = air[U/R^2]$ and so

$$\overline{w}(R)r'[R/R^2] = Sq^{-1}r'[R/R^2] = aiSq^{-1}r'[U/R^2]$$

$$= Sq(ai Sq^{-1}r')[U/R^2]$$

since $\overline{w}(U) = 1$, and this is $Sq a \cdot r'[U/R^2]$. Since the largest nonzero term of $Sq a$ is in dimension $2 \dim(U/R)$, we have $\overline{w}(R)r'[R/R^2] = 0$ if $\dim r' < \dim(R/R^2) - \dim(U/R)$, or $\overline{w}_j(R) = 0$ if $j > \dim(U/R)$.

**COROLLARY.** The classes $\overline{w}_i$ define a homomorphism $H^*BO(m) \to R$.

**PROOF.** We clearly have a homomorphism $F_2[w_1, \ldots, w_m] \to R$ with $w_i$ mapping to $\overline{w}_i$. Projecting into the Poincaré algebra $R/R^2$ for large $s$, the images are Steifel-Whitney classes of a Poincaré algebra and satisfy all relations true in $H^*BO(m)$.

**PROPOSITION.** If $R = H^*BH$ and $S \subset R$ is the subalgebra $H^*BG$ for some compact Lie group $G$ containing $H$, then the Steifel-Whitney class of the adjoint bundle $w(Ad_H)$ restricts to $w(R/S)$ in $R/S$.

**PROOF.** In the fibering $G/H \to BH \to BG$ it is well known that the bundle $Ad_H$ restricts to the normal bundle of the manifold $G/H$. Thus $\pi(w(Ad_H)) = i^*w(Ad_H) \in R/S = H^*G/H$ is the Steifel-Whitney class of the normal Thom module of $R/S$.

Of course, when $R = H^*BH$ there are not enough subalgebras $S \subset R$ of the form $H^*BG$ with $H \subset G$ to make this lead to a proof that $\overline{w}(R) = w(Ad_H)$. By extensive and unpleasant calculation we have verified this equality for all $H$ known to have $H^*BH$ polynomial. It is pointless to reproduce such calculations. Instead we are led to

**PROBLEM.** Does the class $\overline{w} \in R$ defined by using subalgebras coincide with the class $w(Ad_R)$ arising from Hochschild homology?

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