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THE LOCALLY FINITE TOPOLOGY ON 2^X

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ABSTRACT. Let X be a metrizable space. A Vietoris-type topology, called the locally finite topology, is defined on the hyperspace 2^X of all closed, nonempty subsets of X. We show that the locally finite topology coincides with the supremum of all Hausdorff metric topologies corresponding to equivalent metrics on X. We also investigate when the locally finite topology coincides with the more usual topologies on 2^X and when the locally finite topology is metrizable.

1. Introduction. In a recent paper [**HPVV**] the authors proved a measurable selection theorem for closed set-valued maps into complete, but not necessarily separable, metric spaces. The set-valued maps were viewed as point-valued maps into the hyperspace of closed nonempty sets endowed with the Hausdorff metric topology. The measurability assumptions were the usual kinds of measurability of point-valued maps in this setting rather than the more traditional lower or upper measurability (cf. [**H**]). Since the Hausdorff metric topology is not a topological invariant, the measurability of a given set-valued map depended on the particular metric chosen. This is a somewhat less than desirable state of affairs. One way to correct the situation is to endow the hyperspace with the supremum of all Hausdorff metric topologies corresponding to equivalent metrics on the given complete metric space. A result of this paper shows that the supremum topology is actually a Vietoris-type topology, called the locally finite topology, which can be defined on the hyperspace of an arbitrary topological space, not just for a metric space. We also derive others properties of this topology.

Let X be a topological space and let 2^X denote the hyperspace of closed nonempty subsets of X. As is well known, it is possible to topologize 2^X in many different ways. First, when X is metrizable, we can endow 2^X with Hausdorff metric topology. Since we will be dealing with this topology extensively, we introduce the relevant notation. If X is metrized by a metric d, set

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}.$$

Let

$$\vec{\delta}_d(A, B) = \sup\{d(x, B) \mid x \in A\}.$$

The *Hausdorff metric* δ_d, or just δ if d is understood, is given by

$$\delta_d(A, B) = \max\{\vec{\delta}_d(A, B), \vec{\delta}_d(B, A)\}$$

for $A, B \in 2^X$. The topology on 2^X generated by this metric will be denoted by τ_{δ_d}, or more simply τ_δ. (As we have defined δ_d, it can be an infinite-valued metric.)

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It is well known that, in general, equivalent metrics d on X do not determine equivalent Hausdorff metrics δ_d on 2^X. Thus conditions expressed in terms of the topology τ_{δ_d} are not invariant under changes to an equivalent metric ρ on X. A trivial way to avoid this situation is to endow 2^X with a topology τ that is finer than any of the τ_δ's. The smallest such topology is the supremum of all the Hausdorff metric topologies on 2^X. Denote this topology on 2^X by $\tau_{\sup}$. Then

$$\tau_{\sup} = \sup\{\tau_{\delta_d} \mid d \text{ metrizes } X\}.$$

If the metric d metrizes X, then the metric $\rho = \min\{1, d\}$ is uniformly equivalent to d and consequently δ_ρ is equivalent to δ_d. Thus the Hausdorff metric topologies τ_{δ_ρ} and τ_{δ_d} coincide on 2^X and hence the supremum topology $\tau_{\sup}$ on 2^X can be formed by taking the supremum of all the Hausdorff metric topologies corresponding to bounded metrics that give the topology of X. Henceforth we will assume that we are using only bounded metrics on X.

In order to intoduce the next two topologies we need some more notation. If $\mathcal{A}$ is a nonempty family of subsets of X, let

$$\mathcal{A}^- = \{F \in 2^X \mid F \cap A \neq \varnothing \text{ for each } A \in \mathcal{A}\}, \quad \mathcal{A}^+ = \{F \in 2^X \mid F \subseteq \bigcup \mathcal{A}\}.$$

Note that $\mathcal{A}^+ = \{\bigcup \mathcal{A}\}^+$ and so $\mathcal{A}^+$ is usually only needed when $\mathcal{A} = \{A\}$ for some $A \subseteq X$. In this case, i.e., when $\mathcal{A} = \{A\}$, we abuse the notation slightly and write A^+ for $\{A\}^+$ and A^- for $\{A\}^-$.

The *Vietoris* (or *finite*) *topology* τ_{fin} on 2^X has as a subbase all sets of the form U^- and V^+, where U and V range over all open subsets of X. It is easy to check that all sets of the form $V^+ \cap \mathcal{U}^-$, or the form $\mathcal{U}^+ \cap \mathcal{U}^-$, where V is open and $\mathcal{U}$ is a finite family of open subsets of X, are a base for the topology. Basic facts about the Vietoris topology can be found in the fundamental paper of Michael [**Mi**] or the recent monograph of Klein and Thompson [**KT**]. Here we are primarily interested in a finer topology, mentioned by Marjanović [**Ma**] and by Feichtinger [**F**].

DEFINITION. The subbasic open sets for the *locally finite topology* on 2^X are all sets of the form V^+ and $\mathcal{U}^-$, where V ranges over all open subsets of X and $\mathcal{U}$ ranges over the locally finite families of open subsets of X.

We denote the locally finite topology on 2^X by $\tau_{\text{loc fin}}$. Note that $\tau_{\text{fin}} = \tau_{\text{loc fin}}$ if and only if each locally finite collection of open sets in X is finite. (Suppose there exists an infinite, locally finite collection $\mathcal{U}$ of open sets. Then $X \in \mathcal{U}^-$, and each $F \in \mathcal{U}^-$ must be an infinite set, while every $\mathcal{H} \in \tau_{\text{fin}}$ contains finite sets.) In particular, for X paracompact, $\tau_{\text{fin}} = \tau_{\text{loc fin}}$ if and only if X is compact.

In all that follows, if X is a given metrizable space with admissible metric d, we write $B_d(x, \varepsilon)$ for the open ε-ball with center $x \in X$ (omitting d if the metric is understood). We also write $B_d(A, \varepsilon)$ for the union of all ε-balls whose centers run over a subset A of X. As a result, if $\{A, F\} \subseteq 2^X$, then $\delta_d(A, F) = \inf\{\varepsilon \colon A \subseteq B_d(F, \varepsilon) \text{ and } F \subseteq B_d(A, \varepsilon)\}$.

2. The results. In this section we first prove the following theorem.

THEOREM 2.1. *If X is metrizable space, then the locally finite topology on 2^X coincides with the supremum of all the Hausdorff metric topologies corresponding to equivalent metrics on X. That is, $\tau_{\text{loc fin}} = \tau_{\sup}$.*

PROOF. We first show that, for any compatible metric d on X and the corresponding Hausdorff metric $\delta = \delta_d$ on 2^X, $\tau_\delta \subseteq \tau_{\text{loc fin}}$. Given $A \in 2^X$ and $\varepsilon > 0$,

consider a locally finite open cover of X with mesh less than ε, and let $\mathcal{U}$ be the collection of all members of this cover that meet A. Let $V = B_d(A, \varepsilon)$. Then $A \in V^+ \cap \mathcal{U}^- \subseteq B_\delta(A, \varepsilon)$. It follows that $\tau_{\sup} \subseteq \tau_{\text{loc fin}}$.

Conversely, let $\mathcal{U} = \{U_i | i \in I\}$ be any locally finite collection of open sets in X, and consider $A \in \mathcal{U}^+ \cap \mathcal{U}^-$. For each $i \in I$, pick $a_i \in A \cap U_i$. The set $E = \{a_i | i \in I\}$ is discrete, though not necessarily faithfully indexed. For each $e \in E$, set $V_e = \bigcap\{U_i | e \in U_i\} - (E - \{e\})$; note that V_e is an open set with $V_e \cap E = \{e\}$. Form the open cover

$$\{V_e \mid e \in E\} \cup \{U_i - E \mid i \in I\} \cup \{X - A\}$$

of X, and choose a compatible metric ρ on X such that $\{B_\rho(x, 1) \mid x \in X\}$ refines the above cover (cf. Chapter IX, 9.4 of [**D**]). Let $\delta = \delta_\rho$ be its associated Hausdorff metric on 2^X. We claim that $B_\delta(A, 1) \subseteq \mathcal{U}^+ \cap \mathcal{U}^-$. To see this, let $F \in B_\delta(A, 1)$ be arbitrary. Since F meets $B_\rho(e, 1)$ for each $e \in E$ and $B_\rho(e, 1)$ can only lie in V_e, we have $F \in \mathcal{U}^-$. On the other hand, for each $x \in F$, the ball $B_\rho(x, 1)$ must meet A; so, $B_\rho(x, 1) \nsubseteq X - A$. As a result

$$x \in B_\rho(x, 1) \subseteq \left(\bigcup_{e \in E} V_e\right) \cup \left(\bigcup_{i \in I} U_i - E\right) \subseteq \bigcup_{i \in I} U_i,$$

so that $F \in \mathcal{U}^+$ and hence we see that $B_\delta(A, 1) \subseteq \mathcal{U}^+ \cap \mathcal{U}^-$. It follows that $\tau_{\text{loc fin}} \subseteq \tau_{\sup}$.

Next, let $\langle X, d\rangle$ be a metric space. It is well known that the δ_d-topology equals the Vietoris topology iff X is compact. But much more is true (cf. Lemma 3.2 of [**Mi**] and Theorem 2 of [**B**]).

MICHAEL'S LEMMA. *Let $\langle X, d\rangle$ be a metric space. Then $\tau_{\delta_d} \subseteq \tau_{\text{fin}}$ iff X is totally bounded. Moreover, $\tau_{\text{fin}} \subseteq \tau_{\delta_d}$ iff whenever A and B are disjoint closed subsets of X, then* $\inf\{d(a, b) \mid a \in A, b \in B\}$ *is positive.*

Those metric spaces X for which $\inf\{d(a, b) | a \in A, b \in B\}$ is always positive for disjoint closed subsets A and B are precisely those on which each real valued continuous function is uniformly continuous (cf. [**A** or **B**]). For this reason, they are called *UC spaces* in the literature. Evidently a UC space is complete, for if $\langle x_n\rangle$ were a Cauchy sequence in X with distinct terms with no cluster points, the closed sets $\{x_{2_n} \mid n = 1, 2, \dots\}$ and $\{x_{2_{n-1}} \mid n = 1, 2, \dots\}$ would not lie a positive distance apart. Perhaps the most visual internal characterization of a UC space X is this: The set X' of limit points of X is compact, and for each $\varepsilon > 0$, there exists $\delta > 0$ such that whenever x and y are distinct points of $X - B(X', \varepsilon)$, then $d(x, y) > \delta$.

We next consider the relation $\tau_{\text{loc fin}} = \tau_{\delta_d}$ for some compatible metric d on X.

THEOREM 2.2. *Let $\langle X, d\rangle$ be a metric space. Then $\tau_{\delta_d} = \tau_{\text{loc fin}}$ iff $\langle X, d\rangle$ is a UC space.*

PROOF. Suppose $\langle X, d\rangle$ is not a UC space. By Michael's Lemma, τ_{δ_d} fails to contain τ_{fin}, ergo, $\tau_{\text{loc fin}}$. As a result, $\tau_{\delta_d} \neq \tau_{\text{loc fin}}$. Conversely, suppose $\langle X, d\rangle$ is a UC space. Let ρ be another compatible metric. Since id: $\langle X, d\rangle \to \langle X, \rho\rangle$ is continuous, it must be uniformly continuous. As a result, id: $\langle 2^X, \delta_d\rangle \to \langle 2^X, \delta_\rho\rangle$

is uniformly continuous, so that $\tau_{\delta_\rho} \subseteq \tau_{\delta_d}$. Thus, $\tau_{\text{loc fin}} = \tau_{\sup} \subseteq \tau_{\delta_d}$, and we have $\tau_{\delta_d} = \tau_{\text{loc fin}}$.

Theorem 2.2 begs the question: Which metrizable spaces admit UC metrics? This question has been answered in a number of ways by Rainwater [**R**]. Most importantly, X admits a UC metric iff X' is compact. Next we consider the metrizability of $\tau_{\text{loc fin}}$. It turns out that $\tau_{\text{loc fin}}$ is metrizable if and only if X admits a UC metric, and hence by Theorem 2.2, $\tau_{\text{loc fin}}$ is metrizable if and only if it admits a compatible Hausdorff metric.

THEOREM 2.3. *Let X be a metrizable space. The following are equivalent:*
(1) $\tau_{\text{loc fin}}$ *is metrizable.*
(2) $\tau_{\text{loc fin}}$ *is first countable.*
(3) *The set of limit points X' of X is compact.*
(4) *There exists a compatible metric d that makes X a UC space.*
(5) *There exists a compatible metric d for which* $\tau_{\text{loc fin}} = \tau_{\delta_d}$
(6) *There exists a compatible metric d for which* $\tau_{\text{fin}} \subseteq \tau_{\delta_d}$.

PROOF. Theorem 2.2 establishes the equivalence of (4) and (5). Lemma 3.2 of [**Mi**] and Theorem 2 of [**B**] establish the equivalence of (4) and (6). Theorem 2 of [**R**] establishes the equivalence of (3) and (4) (see also [**N**]). The proofs of the implication (5)→(1) and (1)→(2) are trivial. To complete the proof, we establish (2)→(3). Suppose $\tau_{\text{loc fin}}$ is first countable, yet X' is not compact. Then X' contains a countably infinite discrete set $E = \{e_1, e_2, \dots\}$ Choose a discrete collection $\{U_1, U_2, \dots\}$ of open sets in X such that $e_n \in U_n$ for each n. Suppose $\{\mathcal{B}_1, \mathcal{B}_2, \dots\}$ is local base in $\tau_{\text{loc fin}}$ at E. Since each e_n is a limit point of X, we may choose $x_n \in U_n - \{e_n\}$ such that $E \cup \{x_n\} \in \mathcal{B}_n$. Take $V = \bigcup_{n=1}^{\infty}(U_n - \{x_n\})$. Then V is open in X, and $E \in V^+ \in \tau_{\text{loc fin}}$, so for some n, $\mathcal{B}_n \subseteq V^+$. But this impossible, since $E \cup \{x_n\} \notin V^+$.

It should be noted that 2^X equipped with $\tau_{\text{loc fin}}$ is always a uniform space (and thus is Tychonoff) whenever X is metrizable. Specifically, if $\{d_i | i \in I\}$ are the compatible metrics for X, then a base for a diagonal uniformity compatible with $\tau_{\text{loc fin}}$ (cf. [**D**]) consists of all sets in $2^X \times 2^X$ of the form

$$\left\{(A, B) \mid \max_{i \in J} \delta_{d_i}(A, B) < 1/n\right\},$$

where $J \subset I$ is finite and n is a positive integer.

Finally, we note a fact that should be apparent from the proofs of Theorems 2.1 and 2.3: If X is metrizable, then the locally finite topology on 2^X has as subbasic open sets all sets of the form V^+ and $\mathcal{U}^-$, where V ranges over all open subsets of X and $\mathcal{U}$ ranges over the discrete families of open subsets of X.

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