PRIMES DIVIDING CHARACTER DEGREES
AND CHARACTER ORBIT SIZES

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Abstract. We consider an abelian group $A$ which acts faithfully and coprimely on a solvable group $G$. We show that some $A$-orbit on $\text{Irr}(G)$ must have cardinality divisible by almost half the primes in $\pi(A)$. As a corollary, we improve a recent result of I. M. Isaacs concerning the maximum number of primes dividing any one character degree of a solvable group.

A recent result of I. M. Isaacs [5, Corollary 4.3] relates the maximum number of primes dividing any one irreducible character degree of a solvable group $G$ to the number of primes dividing all the character degrees of $G$ taken together. Here we considerably strengthen the bound in [5, Corollary 4.3] by proving a result on character orbit sizes in coprime actions.

We consider an abelian group $A$ which acts faithfully and coprimely on a solvable group $G$. We show that some $A$-orbit on $\text{Irr}(G)$ must have cardinality divisible by almost half the primes in $\pi(A)$. Our approach roughly parallels that of [6]. This paper and [5] contain new applications of the results and methods of [6, 7, 1, 2].

I would like to thank I. M. Isaacs for bringing this problem to my attention.

Our notation is largely standard. If a group $G$ acts on a set $\Omega$ and $\omega \in \Omega$, we denote by $\text{Orb}_G(\omega)$ the $G$-orbit of $\omega$. If $G < H$ and $H$ also acts on $\Omega$, we say that $h \in H$ moves $\text{Orb}_G(\omega)$ if $\omega^h \not\in \text{Orb}_G(\omega)$. All groups considered in this paper are finite and solvable.

We now state our main results.

Theorem 1. Let $A$ act faithfully on $G$ with $(|A|, |G|) = 1$ and $A$ abelian. Then $|\pi(A)| \leq 2|\pi(\text{Orb}_A(\chi))| + 8$ for some $\chi \in \text{Irr}(G)$.

Corollary 1. Let $G$ be solvable. Let $s = \max\{|\pi(\chi(1))|: \chi \in \text{Irr}(G)\}$ and let $\rho = |\pi(\Pi_{\chi \in \text{Irr}(G)}\chi(1))|$. Then $\rho \leq s^2 + 10s$.

Proof. Using Theorem 1 above, we get a stronger version of [5, Theorem 4.1] in which part (a) is replaced by $|\rho(G) - \rho(N) - \sigma| \leq 2(s - |\sigma|) + 8$. This leads to a stronger version of [5, Corollary 4.3] in which $|\rho(G)| \leq s + \sum_{i=0}^{s-1}2(s - i) + 8 = s^2 + 10s$. Since $|\rho(G)|$ in [5] is called $\rho$ in our paper, this completes the proof.
The next proposition shows that the bound in Theorem 1 is close to best possible.

**Proposition 1.** Let $m$ be a positive integer. There exist groups $A$ and $G$ satisfying the hypotheses of Theorem 1 with $|\pi(A)| = 2m$ and $\pi(\text{Orb}_A(\chi)) \leq m$ for all $\chi \in \text{Irr}(G)$.

**Proof.** For $1 \leq i \leq m$, choose odd primes $p_i$ and $q_i$ subject to the following conditions. Let $e_i$ and $f_i$ denote the order of 2 mod $p_i$ and $q_i$ respectively.

1. \(\min(p_i, q_i) > 2^{p_i-1}q_i - 1\) for $i > 1$,
2. \(q_i = 2^{e_i} \cdot \prod \text{odd primes} \leq q_i\) for $i > 1$,
3. \(f_i < e_i\) for all $i$.

For each $i$ set $n_i = f_i p_i$. Clearly $2^{f_i} - 1 \equiv 0 \pmod{q_i}$. Since

\[ (2^{n_i} - 1)/(2^{f_i} - 1) \equiv 1 + 2^{f_i} + \cdots + 2^{(p_i-1)f_i}, \]

it follows that $(2^{n_i} - 1)/(2^{f_i} - 1) \equiv p_i \not\equiv 0 \pmod{q_i}$. By (3) above, $2^{n_i} - 1 = 2^{f_i}p_i - 1 \equiv 2^{f_i} - 1 \not\equiv 0 \pmod{p_i}$.

Let $V_i$ be elementary abelian of order $2^{n_i}$. Let $C_i$ be cyclic of order $2^{n_i} - 1$. Let $P_i$ be cyclic of order $p_i$, and let $P_i$ act on $C_i \cong \text{GF}(2^{n_i} - 1)^\times$ as the subgroup of order $p_i$ in $\text{Gal}(\text{GF}(2^{n_i}))/\text{GF}(2))$. Let $H_i = (P_i, C_i) V_i$ be the corresponding subgroup of the affine semilinear group over $\text{GF}(2^{n_i})$. Let $Q_i$ be the Sylow $q_i$-subgroup of $C_i$. The preceding paragraph implies that $[P_i, Q_i] = 1$ and $Q_i > 1$. Let $C_i = Q_i \times R_i$. Let $G_i = R_i V_i$ and $A_i = P_i Q_i$. Then $A_i$ acts faithfully on $G_i$ and the preceding paragraph shows that $(|A_i|, |G_i|) = 1$.

Let $\chi \in \text{Irr}(H_i)$ and let $\lambda$ be an irreducible constituent of $\chi V_i$. If $\lambda = 1$, then $\chi(1)$ divides $p_i$. If $\lambda \neq 1$, then $|H_i(\lambda)| = p_i$, and every character in $\text{Irr}(H_i(\lambda)|\lambda)\lambda)$ has degree 1 (see [4, 6.17 and 6.20]). Hence $\chi(1) = |Q_i| |R_i|$. In either case, at most one prime in $\pi(A_i)$ divides $\chi(1)$.

Let $\psi \in \text{Irr}(G_i)$. Since $G_i < H_i$, $|\text{Orb}_A(\psi)|$ divides an irreducible character degree of $H_i$. Hence $|\pi(\text{Orb}_A(\psi)))| \leq 1$.

Now let $G = G_1 \times \cdots \times G_m$ and let $A = A_1 \times \cdots \times A_m$ act componentwise on $G$. Clearly $A$ acts faithfully on $G$. To show that $(|A|, |G|) = 1$, it suffices, by induction, to show that $(|A_i|, |G_i|) = (|A_i|, |G_m|) = 1$ for $i < m$. We already know that $(|A_m|, |G_m|) = 1$. Suppose $i < m$. If $(|A_m|, |G_i|) > 1$, then $(|A_m|, |C_i|) = (|A_m|, 2^{p_i} - 1 > 1$. Since $f_i$ divides $q_i - 1$, this contradicts condition (1) in the first paragraph. If $(|A_i|, |G_m|) > 1$ for $i < m$, then $(|A_i|, |C_m|) = (|A_i|, 2^{p_i m} - 1 > 1$. Then ord$_{(2)}|p_i f_m$ where $r = p_i$ or $r = q_i$. Since ord$_{(2)}$ divides $r - 1 < p_m$, we have ord$_{(2)}|f_m$ and so ord$_{(2)}|(q_m - 1)$, contrary to condition (2) in the first paragraph. Hence $A$ and $G$ satisfy the hypothesis of Theorem 1.

Let $\psi = \psi_1 \times \cdots \times \psi_m$ be an arbitrary character in $\text{Irr}(G)$. As above, we may choose $S_i \in \{P_i, Q_i\}$ so that $S_i$ fixes $\psi_i$. Then $S_1 \times \cdots \times S_m$ fixes $\psi$, so $|\pi(\text{Orb}_A(\psi)))| \leq m$. This completes the proof.

We proceed to prove Theorem 1. We will use the proof of [6, Theorem 3.3] as a rough guide in the proof of Proposition 3 below.

**Lemma 1.** Let $A$ act on $G$ with $A$ abelian and $(|A|, |G|) = 1$. Let $A_p$ denote the $p$-Sylow subgroup of $A$. Let $N$ and $M$ be $A$-invariant normal subgroups of $G$, with $[A_p, G] \leq M$ and $N \leq M$. Let $\lambda$ be an irreducible character of $N$ and suppose $A_p$
moves \( \text{Orb}_M(\lambda) \). Then \( A_p \) moves \( \text{Orb}_G(\lambda) \). Similarly, if \( v \in N \) and \( A_p \) moves \( \text{Orb}_M(v) \), then \( A_p \) moves \( \text{Orb}_G(v) \).

**Proof.** Suppose \( A_p \) stabilizes \( \text{Orb}_G(\lambda) \). Then the semidirect product \( A_pG \) acts on \( \text{Orb}_G(\lambda) \) with \( G \) acting transitively. By Glauberman's Lemma [4, Lemma 13.8], \( A_p \) fixes some \( \psi \in \text{Orb}_G(\lambda) \). Then \( A_p^g \) fixes \( \lambda \) for some \( g \in G \). Since \( G = C_G(A_p)|A_p, G| = C_G(A_p)M \), we may assume that \( g \in M \). Hence \( A_p \) stabilizes \( \text{Orb}_M(\lambda) \).

The second assertion is proved similarly.

**Lemma 2.** Let \( G \neq 1 \) be solvable with every normal abelian subgroup cyclic. Let \( p_1, \ldots, p_n \) be the distinct prime divisors of \( |F(G)| \) and let \( Z < Z(F(G)) \) with \( |Z| = p_1 \cdots p_n \). Let \( D = C_G(Z) \). Then there exist \( E, T < G \) with

(i) \( ET = F(G) \) and \( E \cap T = Z \).

(ii) Each Sylow subgroup of \( T \) is cyclic, dihedral, semidihedral or quaternion.

(iii) \( T \) has a cyclic subgroup \( U \) with \( |T : U| \leq 2 \) and \( U < G \).

(iv) Each Sylow subgroup of \( E \) is cyclic of prime order or extraspecial of prime exponent or exponent 4.

(v) \( G \) is nilpotent if and only if \( G = T \).

(vi) \( T = C_G(E) \) and \( F(G) = C_D(E/Z) \).

(vii) Each Sylow subgroup of \( E/Z \) is elementary abelian and is a completely reducible \( D/F(G) \)-module.

**Proof.** This is [7, Corollary 2.4].

**Lemma 3.** Let \( G, E, U, \) and \( Z \) be as in Lemma 2. Let \( V \) be a faithful \( F[EU] \)-module for a finite field \( F \). Let \( W \neq 0 \) be an irreducible \( U \)-submodule of \( V \) and let \( e = |E : Z|^{1/2} \). Then \( \dim V = me \dim W \) for an integer \( m \).

**Proof.** This is [7, Lemma 2.5].

**Lemma 4.** Let \( E \triangleleft H \) with \( [H : E] = p \) and \( p \nmid |E| \). Let \( Z = Z(E), \) \( P \in \text{Syl}_p(H), \) and let \( V \) be a finite-dimensional \( F[H] \)-module for a field \( F \). Assume that \( E/Z \) is an abelian \( q \)-group for a prime \( q, P \not\in C_H(E), \) and \( V_E \) is a faithful, completely reducible and homogeneous module. Then \( \dim C_V(P) \leq (\dim V)/2 \) if \( p \) is odd.

**Proof.** This is part of [6, Lemma 1.7].

**Lemma 5.** Let \( V \neq 0 \) be a faithful and completely reducible \( F[G] \)-module for a field \( F \) and a solvable group \( G \). Then \( |G| \leq |V|^{9/4} \).

**Proof.** This is a slightly weaker version of [7, Theorem 3.1].

**Proposition 2.** Let \( A \) act on \( G \) with \( (|A|, |G|) = 1 \) and \( A \) cyclic of squarefree order. Suppose that \( [A_p, G/F(G)] \) is a nonidentity abelian group for all \( p \) in \( \pi(A) \). Then \( A \) has a faithful orbit on \( \text{Irr}(G) \).

**Proof.** Let \( H = G/F(G) \). Then \( [A, H] = \prod_{p \in \pi(A)} [A_p, H] \) is contained in \( F(H) \). Let \( W = [A, H]/\Phi([A, H]) \), so that \( W \) is a direct product of elementary abelian \( q \)-groups for primes \( q \) dividing \( |H| \). Write \( W = W_1 \times \cdots \times W_k \), each \( W_i \) an irreducible \( A \)-module. For \( 1 \leq i \leq k \), let \( 1 \neq \lambda_i \in \text{Irr}(W_i) \). Let \( \lambda = \lambda_1 \times \cdots \times \lambda_k \). Since...
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(|A|, |H|) = 1, A acts faithfully on [A, H] and hence on W. For each \( p \in \pi(A/C_\lambda(W_i)) \), thus \( \lambda_i \) is moved by \( A_p \) for every \( p \in \pi(A/C_\lambda(W_i)) \). Thus \( \lambda \) lies in a faithful \( A \)-orbit on \( \text{Irr}(W) \). We now apply Lemma 1 with \( A, H, [A, H], [A, H] \) in place of \( A, G, M, N \) and conclude that \( \text{Orb}_G(\lambda) \) is moved by \( A_p \) for every \( p \in \pi(A) \). Let \( \chi \in \text{Irr}(H|\lambda) \).

Then \( \chi \) lies in a faithful \( A \)-orbit on \( \text{Irr}(H) \subseteq \text{Irr}(G) \).

**Proposition 3.** Let \( \pi_0 = \{2, 3, 5, 7, 11, 13, 17, 31\} \). Let \( A \) be cyclic of squarefree \( \pi_0 \)-order. Let \( A \) act on \( G \) with \( (|A|, |G|) = 1 \) and \( [A_p, G] \) nonabelian for all \( p \in \pi(A) \). Let \( V \) be an abelian group which is a direct product of completely reducible \( AG \)-modules over various finite fields. Suppose \( (|A|, |V|) = 1 \) and \( AG \) acts faithfully on \( V \). Then there exists \( v \in V \) such that \( \text{Orb}_G(v) \) is moved by every \( A_p \).

**Proof.** We proceed by induction on \( |G| + |V| \). Set \( \pi(A) = \pi \).

First suppose \( V \) is not an irreducible \( AG \)-module. Write \( V = V_1 \times \cdots \times V_k \), with each \( V_i \) an irreducible \( AG \)-module. Let \( G_i = G/C_G(V_i) \) for \( 1 \leq i \leq k \). Let \( \pi_i = \{ p \in \pi : [A_p, G_i] \text{ is nonabelian} \} \). For each \( p \in \pi \), \( [A_p, G] \) is isomorphic to a subgroup of \( [A_p, G_1] \times \cdots \times [A_p, G_k] \). Hence \( [A_p, G_i] \) is nonabelian for some \( i \), and so \( \pi = \pi_1 \cup \cdots \cup \pi_k \). We apply the induction hypothesis to \( G_i \) with \( \pi_p \) in place of \( A, G, V, V \). We obtain \( v_i \in V_i \) such that \( \text{Orb}_{G_i}(v_i) \) is moved by \( A_p \) for all \( p \in \pi \). Thus \( \text{Orb}_G(v_1, \ldots, v_k) \) is moved by \( A_p \) for all \( p \in \pi \).

We now assume that \( V \) is an irreducible \( AG \)-module. We may apply Lemma 1 with \( A, G, V, [A, G]V, V \) in place of \( A, G, M, N \) and conclude that it suffices to find \( v \in V \) such that \( \text{Orb}_{[A, G]}(v) \) is moved by \( A_p \) for all \( p \in \pi \). By the inductive hypothesis we may then assume that \( G = [A, G] \). It follows that \( O^{\pi}(AG) = AG \), since otherwise a proper factor group of \( G \) would be centralized by \( A \), contrary to \( G = [A, G] \).

Suppose that \( V \) is imprimitive. Let \( V = V_1 \oplus \cdots \oplus V_t \) be an imprimitivity decomposition for the action of \( AG \) on \( V \). We may partition \( \{1, \ldots, t\} \) into blocks \( B_j \), \( 1 \leq j \leq s \), and set \( U_j = \Sigma_{i \in B_j} V_i \), so that \( AG \) permutes the set \( \{U_1, \ldots, U_s\} \) primitively. Let \( C \) be the kernel of the permutation action of \( AG \) on the \( U_j \). Since \( AG = O^\pi(AG) \), we have \( A \not\subseteq C \). By [1, Theorem 1] we may choose \( S \leq \{1, 2, \ldots, s\} \) so that the stabilizer in \( AG \) of \( \{U_j \mid j \in S\} \) is \( C \). Let \( U = \Sigma_{j \in S} U_j \). Let \( \pi_1 = \{ p \in \pi : [A_p, G \cap C] \subseteq \} \). Let \( A_1 = \prod_{p \in \pi_1} A_p \) so that \( C = A_1(C \cap G) \). By [1, Theorem 1] the irreducible constituents of \( V_{C \cap G} \) are \( AG \)-conjugate. Thus if \( K_j \) denotes the kernel of \( C \) on \( U_j \), then \( \cap_{x \in AG} K_j^x = 1 \) for each \( j \in S \). Let \( p \in \pi_1 \). Since all \( p \)-Sylow subgroups of \( C \) are conjugate under \( G \cap C \), it follows that \( [A_p, G \cap C] \cap G \cap C \) and \( [A_p, G \cap C]' \cap G \cap C \). The last two sentences imply that \( [A_p, G \cap C]' \not\subseteq K_j \) for any \( j \in S \). Hence \( A_p \not\subseteq K_j \) for \( p \in \pi_1 \) and \( j \in S \). Since \( K_j \not\subseteq \), it follows that \( K_j \not\subseteq G \cap C \) and \( [A_p, (G \cap C)/K_j] \) is nonabelian for \( p \in \pi_1 \) and \( j \in S \).

For \( j \in S \), we may now apply the inductive hypothesis with \( A_1, C \cap G/K_j, U_j \) in place of \( A, G, V \). We obtain \( u_j \in U_j \) such that \( \text{Orb}_{C \cap G}(u_j) \) is moved by \( A_p \) for all \( p \in \pi_1 \). Let \( u = \Sigma_{j \in S} u_j \). If \( p \in \pi_1 \) and \( p \in \text{Syl}_p(AG) \), then the choice of \( S \) insures that \( P \) does not centralize \( u \). If \( P \in \text{Syl}_p(AG) \) and \( p \in \pi_1 \), then the
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The definition of \( u_j \) implies that \( P \) does not centralize \( u \). For every \( p \in \pi \), the last two sentences show that \( A_p \) fixes no element in \( \text{Orb}_C(u) \). If \( A_p \) stabilized \( \text{Orb}_C(u) \), then Glauberman’s Lemma applied to the action of \( A_p G \) on \( \text{Orb}_C(u) \) would yield a contradiction. Hence \( A_p \) moves \( \text{Orb}_C(u) \) as desired.

We may now assume that \( V \) is a primitive \( AG \)-module. If \( F(AG) \not\subseteq G \), then some \( A_p \leq F(AG) \), and so \( A_p \not\leq AG \), contrary to \([A_p, G] \neq 1\). Hence \( F(AG) \subseteq G \), and so \( F(AG) = F(G) \). Set \( F(AG) = F \). Now \( AG \) can play the role of “\( G \)” in Lemma 2. Let \( T, U \leq AG \) be as in the conclusion of Lemma 2. Suppose \( T \neq U \). Then every \( 2' \)-element of \( AG \) centralizes \( O_2(T) \), so \( AG/CAG(O_2(T)) \) is a nonidentity \( 2 \)-group, contradicting \( O^2(AG) = AG \). Thus \( T = U \) is cyclic. Let \( Z, D, \) and \( E \) be as in Lemma 2, so that \( F = C_p(E/Z) \) and each Sylow subgroup of \( E/Z \) is a completely reducible \( D/F \)-module.

Fix \( p \in \pi \). Since \( AG/D \) is abelian, \([A_p, G] \leq D \cap G \). Thus \([A_p, G] = [A_p, A_p, G] = [A_p, D \cap G] \). Suppose \([A_p, E/Z] = 1\). Since \( D \) and \( C_{AG}(E/Z) \) are normal in \( AG \), we have \([A_p, G] = [A_p, D \cap G] \leq C_p(E/Z) = F \). Hence

\[
[A_p, G] = [A_p, A_p, G] = [A_p, F] = [A_p, E] [A_p, T] \leq ZT = T,
\]

contary to the hypotheses of Proposition 3. Hence \([A_p, E/Z] \neq 1\). Let \( E_1 \) be a Sylow subgroup of \( E \) with \([A_p, E_1 \cap E_1 \cap Z] \neq 1\). We apply Lemma 4 to \( A_p E_1, E_1, A_p, V \) in place of \( H, E, P, V \). We conclude that \(|C_{E}(A_p)| \leq |V|^{1/2}\).

Let \( Y \) be an irreducible \( F \)-submodule of \( V \). By Lemma 3, \(|Y| = |W|^{me} \), where \( e^2 = |E:Z| \) and \( m \) is a positive integer. Moreover \( W \) is a faithful irreducible \( T \)-submodule of \( Y \), so that \(|T| \) divides \(|W| - 1\).

Now \(|G \cap D| = |G \cap D:F| |F| \). By Lemmas 2 and 5, and an obvious subdirect product argument, \( |D:F| \leq |E:Z|^{1/4} = e^{1/2} \). We have \(|G \cap D| \leq e^{1/2} |e/2| |T| = |T|^e^{1/2} \). Since \([A_p, G] = [A_p, G \cap D] \), we have \( O^p(AG) \leq A_p(G \cap D) \), so \( |Syl_p(AG)| \leq |G \cap D| \leq e^{1/2} |T| \). Hence

\[
\sum_{p \in Syl_p(AG)} |C_{E}(p)| \leq |T| |T|^{e^{1/2}} |C_{E}(A_p)| \leq |T| |T|^{e^{1/2}} |V|^{1/2}.
\]

We will show that the following inequality holds:

\[
(*) \sum_{p \in Syl_p(AG)} |C_{E}(p)| \leq p^{-2} |V|.
\]

Suppose (*) is false, so that \( p^2 |T| |T|^{e^{1/2}} \geq |V|^{1/2} \geq |W|^{e/2} \).

If \([A_p, Z] \neq 1\), then \( p \) divides \(|\text{Aut } Z|\) and so \( p |(s - 1) \) for some prime divisor \( s \) of \(|Z|\). Since \( Z = T \cap E \), we have \( s |e \) and \( s \leq |T| < |W| \). Since \( p > 17 \) and \( p |(s - 1) \), it follows that \( s \geq 47 \). Since \( p < s \leq |T| < |W| \), we have \(|W|^{3e^{1/2}} > |W|^{e/2} \), so that \( e^{1/2} > |W|^{(e/2)^{-3}} > 48^{(e/2)^{-3}} \). Hence \( e < 20 \), contrary to \( s \geq 47 \) and \( s \leq e \).

Thus we assume \([A_p, Z] = 1\). Let \( e = \prod_i q_i^{n_i} \), for distinct primes \( q_i \). Since \([A_p, E/Z] \neq 1 \) and \( A_p \not\leq D \), \( p \) divides \(|\text{Sp}(2n_i, q_i)|\) for some \( i \). Hence \( p | q_i^{m_i} - 1 \) for some \( m_i \) with \( 1 \leq m_i \leq n_i \). Thus \( p | q_i^{m_i} + 1 \) or \( p | q_i^{m_i} - 1 \), where \( q_i^{m_i} | e \). It follows that \( p \leq e + 1 \). Since \( |T| < |W| \), we have \((e + 1)^{e^{1/2}} > |W|^{(e/2)^{-1}} \). Since \(|W| \geq 3 \), we have \( e < 70 \).
If \( m_i > 1 \), our hypothesis that \( p \not\in \pi_0 \) implies that \( q_i^{m_i} > 70 \). Hence \( e > 70 \), a contradiction. Thus \( m_i = 1 \) and \( p \not\in \pi_0 \) implies that \( q_i \geq 37 \). Since \( q_i \) divides \( |T| \) and \( |T| \) divides \( |W| - 1 \) and \( |W| \) is a prime power, we must have \( |W| \geq 83 \). Hence \((e + 1)^2 e^{13/2} > 83^{(e/2)-1}\). As above, this implies that \( e < 20 \), contrary to \( q_i | e \) and \( q_i \geq 37 \).

We conclude that \((*)\) holds for each \( p \in \pi \). Since \( \Sigma p \in \pi p^{-2} \) is less than 1, it follows that there exists \( v \in V \) such that \( v \) is centralized by no \( p \)-Sylow subgroup of \( AG \) for any \( p \in \pi \). Then no \( A_p \) fixes any element in \( \text{Orb}_G(v) \). By Glauberman’s Lemma, each \( A_p \) moves \( \text{Orb}_G(v) \).

**Proposition 4.** Let \( A \) be cyclic of squarefree \( \pi_0 \)-order. Let \( N \) be an \( A \)-invariant normal abelian subgroup of \( G \) which is a direct product of completely reducible \( AG \)-modules. Suppose \( N = C_{AG}(N) \). For each \( p \in \pi(A) \), suppose that \([A_p, G/N]\) is either nonabelian or trivial. Then \( A \) has a faithful orbit on \( \text{Irr}(G) \).

**Proof.** Let \( V = \text{Irr}(N) \), \( \pi = \pi(A) \), \( \pi_1 = \{ p \in \pi : [A_p, G/N] \text{ is nonabelian} \} \), and \( \pi_2 = \pi - \pi_1 \). Let \( A_1 \) be the Hall \( \pi_1 \)-subgroup of \( A \).

Then \( A_1 \), \( G/N \), and \( N \) satisfy the hypotheses of Proposition 3. Hence we may choose \( v \in V \) so that \( \text{Orb}_G(v) \) is moved by \( A_p \) for all \( p \in \pi_1 \).

Write \( V = V_1 \times \cdots \times V_k \), a direct product of irreducible \( AG \)-modules. We may assume that each component \( V_i \) of \( v \) is not 1. For each \( p \in \pi_2 \), we may choose \( i \in \{1, \ldots, k\} \) such that \( V_i \) is not centralized by \( A_p \). Since \([A_p, G] \leq N = C_{AG}(V)\), the centralizer in \( V_i \) of \( A_p \) is an \( AG \)-submodule of \( V_i \), and hence is trivial. Thus \( A_p \) moves \( V_i \), and so \( A_p \) moves \( v \).

We think of \( v \) as a linear character \( \lambda \) of \( N \). Then \( \text{Orb}_N(\lambda) = \{\lambda\} \) is not \( A_p \)-invariant for any \( p \in \pi_2 \). By Lemma 1 with \( A, G, M, N \) in place of \( \pi, G, M, N \), we conclude that \( \text{Orb}_G(\lambda) \) is moved by \( A_p \) for all \( p \in \pi_2 \). By the second paragraph, \( \text{Orb}_G(\lambda) \) is moved by \( A_p \) for all \( p \in \pi \). Hence if \( \chi \in \text{Irr}(G|\lambda) \), then \( \chi \) lies in a faithful \( A \)-orbit.

**Proof of Theorem 1.** We may assume \( A \) is cyclic of squarefree order. The hypotheses of Theorem 1 imply that \( F(AG) = F(G) \). Since \( G = [A_p, G]C_{AG}(A_p) \) for each \( p \in \pi(A) \), it follows that \( A \) acts faithfully on \( G/F(AG) \). Thus we may assume that \( \Phi(AG) = 1 \) and hence that \( F(G) = F(AG) \) is a direct product of completely reducible \( AG \)-modules (see [3, III, Satz 4.5]).

Partition \( \pi(A) = \pi \) as follows. Let \( \pi_1 = \pi \cap \pi_0 \), \( \pi_2 = \{ p \in \pi : p \not\in \pi_0 \} \), and \( [A_p, G/F(G)] \) is abelian and nontrivial}, \( \pi_3 = \pi - \pi_1 - \pi_2 \), and \( \pi_4 \) be the larger of \( \pi_2 \) and \( \pi_3 \). Let \( A_4 \) be the Hall \( \pi_4 \)-subgroup of \( A \). By Proposition 3 or Proposition 4 applied to \( A_4 \) and \( G \), we may conclude that \( A_4 \) has a faithful orbit on \( \text{Irr}(G) \). Since \(|\pi(A)| \leq 2|\pi(A_4)| + 8\), this completes the proof.

**References**


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