

## REDUCTION EXPONENT AND DEGREE BOUND FOR THE DEFINING EQUATIONS OF GRADED RINGS

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ABSTRACT. The paper gives upper degree bounds for the defining equations of certain graded rings in terms of the reduction exponent and the multiplicity.

**1. Introduction.** Let  $(A, \mathfrak{m})$  be a local ring and  $\mathfrak{a}$  an  $\mathfrak{m}$ -primary ideal. Let  $G_{\mathfrak{a}}(A)$  denote the associated graded ring  $\bigoplus_{n=0}^{\infty} \mathfrak{a}^n / \mathfrak{a}^{n+1}$ . Hong [7], Cho [2], and Achilles and Schenzel [1] have shown that if  $A$  is a one-dimensional Buchsbaum (resp. Cohen-Macaulay) ring, then the degrees of the defining equations of  $G_{\mathfrak{a}}(A)$  represented as a quotient of a polynomial ring over  $A/\mathfrak{a}$  are bounded above by  $e(A) + 1$  (resp.  $e(A)$ ), where  $e(A)$  denotes the multiplicity of  $A$  with respect to  $\mathfrak{m}$ . The main goal of this paper is to generalize this result for higher-dimensional Buchsbaum (resp. Cohen-Macaulay) rings. Moreover, we also want to explain the phenomenon that all the above works employed, in nonapparent and different ways, the reduction exponent of a minimal reduction of  $\mathfrak{a}$  and obtained its invariance as a by-product. This is of some interest because Sally [13] has discussed the problem whether  $A$  being Cohen-Macaulay implies the invariance of the reduction exponent of minimal reductions of  $\mathfrak{m}$ . Recall that an ideal  $\mathfrak{b} \subseteq \mathfrak{a}$  is called a reduction of  $\mathfrak{a}$  if  $\mathfrak{a}^{n+1} = \mathfrak{b}\mathfrak{a}^n$  for some nonnegative integer  $n$  [9] and that the reduction exponent of  $\mathfrak{b}$  is the least integer  $n$  with this property [11, 12]. (For the theory of Buchsbaum rings see [14].)

To achieve the above goal we will study the relationship between the reduction exponent  $r(I)$  of a minimal reduction  $I$  of the positively graded part of a graded ring  $R = \bigoplus_{n=0}^{\infty} R_n$  and the degrees of the defining equations of  $R$  represented as a quotient of a polynomial ring over  $R_0$ . It will turn out that these degrees are bounded above by  $\max\{r(I) + 1, a(I)\}$ , where  $a(I)$  denotes the least integer  $n$  such that  $I$  can be generated by a sequence of elements which is regular in degree  $\geq n$ . This bound is independent of the choice of  $I$  because it can be expressed by means of the local cohomology modules of  $R$  in a manner like Castelnuovo's regularity (recently studied by Ooishi [10], and Eisenbud and Goto [4]). If  $I$  can be generated by a  $d$ -sequence [8], we always have  $r(I) + 1 \geq a(I)$ . This implies the invariance of  $r(I)$  and the degree bound  $r(I) + 1$  for the defining equations of a graded Buchsbaum ring  $R$  with  $R_0$  being a zero-dimensional local ring. In particular, by comparing  $r(I)$  with the multiplicity  $e(R)$  of  $R$  with respect to the maximal graded ideal, we shall obtain the following result.

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**THEOREM 1.1.** *Let  $R$  be a graded Buchsbaum (resp. Cohen-Macaulay) ring with  $R_0$  a zero-dimensional local ring. Then the degrees of the defining equations of  $R$  are bounded above by  $e(R) + 1$  (resp.  $e(R)$ ).*

A modified version of the above method applied to  $G_{\mathbf{a}}(A)$  will yield the following generalization of the results of Hong, Cho, Achilles, and Schenzel.

**THEOREM 1.2.** *Let  $A$  be a Buchsbaum (resp. Cohen-Macaulay) ring with  $d = \dim(A) \geq 1$  and  $\text{depth}(G_{\mathbf{a}}(A)) \geq d - 1$ . Let  $r(\mathbf{b})$  denote the reduction exponent of a minimal reduction  $\mathbf{b}$  of  $\mathbf{a}$ . Then*

- (i)  $r(\mathbf{b})$  is independent of the choice of  $\mathbf{b}$ ,
- (ii) the degrees of the defining equations of  $G_{\mathbf{a}}(A)$  are bounded above by  $r(\mathbf{b}) + 1$ ,
- (iii)  $r(\mathbf{b}) \leq e(A)t^{d-1}$  (resp.  $e(A)t^{d-1} - 1$ ), where  $t$  is the least integer with the property  $\mathbf{m}^t \subseteq \mathbf{a}$ .

We would like to mention that a special situation of the Cohen-Macaulay case of Theorem 1.2 was already considered by Cho [2] who claimed that if moreover  $G_{\mathbf{a}}(A)$  is free over  $A/\mathbf{a}$ , the degrees of the defining equation of  $G_{\mathbf{a}}(A)$  are bounded above by  $e(A)$ . But the proof for that contains an error which could not be corrected if  $d > 1$  (see Remark 5.2).

The proofs of the above theorems will be found in §§4 and 5, respectively. §§2 and 3 only have a preparatory character.

**2. Filter-regular sequence.** Throughout this paper, let  $R = \bigoplus_{n=0}^{\infty} R_n$  be a noetherian commutative graded ring such that  $R_0$  is a local ring and the  $R_0$ -algebra  $R$  is generated by the elements of  $R_1$ . Let  $M$  denote the unique maximal graded ideal of  $R$  and  $k$  the residue field  $R/M$ . Put  $R^+ = \bigoplus_{n \geq 1} R_n$ . If  $E$  is a graded module over  $R$ , then we denote by  $E_n$  the  $n$ th graded piece of  $E$ .

Let  $\mathbf{f} = f_1, \dots, f_r$  be a sequence of homogeneous elements of  $R$ ,  $r \geq 1$ . One can use the following notion to measure how much  $\mathbf{f}$  differs from being a regular sequence.

**DEFINITION.**  $\mathbf{f}$  is called  $n$ -regular if

$$((f_1, \dots, f_{i-1}) : f_i)_n = (f_1, \dots, f_{i-1})_n$$

for  $i = 1, \dots, r$ . The least integer  $m$  such that  $\mathbf{f}$  is  $n$ -regular for all  $n \geq m$  will be denoted by  $a(\mathbf{f})$ .

We shall see that the condition  $a(\mathbf{f}) < \infty$  can be characterized by the behavior of  $f_i$  towards the associated primes of  $(f_1, \dots, f_{i-1})$ ,  $i = 1, \dots, r$ .

**DEFINITION.**  $\mathbf{f}$  is called filter-regular (with respect to  $R^+$ ) if  $f_i \notin P$  for all primes  $P \in \text{Ass}(R/(f_1, \dots, f_{i-1}))$ ,  $P \not\subseteq R^+$ ,  $i = 1, \dots, r$ .

This notion has its origin in the theory of Buchsbaum rings [3] and is closely related to the notion of  $d$ -sequences [15].

**LEMMA 2.1.**  $a(\mathbf{f}) < \infty$  if and only if  $\mathbf{f}$  is filter-regular.

**PROOF.**  $((f_1, \dots, f_{i-1}) : f_i)_n = (f_1, \dots, f_{i-1})_n$  for  $n$  sufficiently large if there exists an integer  $m$  such that

$$(R^+)^m((f_1, \dots, f_{i-1}) : f_i) \subseteq (f_1, \dots, f_{i-1})$$

or, equivalently,

$$(f_1, \dots, f_{i-1}) : f_i \subseteq \bigcup_{n=0}^{\infty} (f_1, \dots, f_{i-1}) : (R^+)^n.$$

Since  $\bigcup_{n=0}^{\infty} (f_1, \dots, f_{i-1}) : (R^+)^n$  is the intersection of all primary components of  $(f_1, \dots, f_{i-1})$  whose associated primes do not contain  $R^+$ , the last inclusion is satisfied iff  $f_i \notin P$  for all primes  $P \in \text{Ass}(R/(f_1, \dots, f_{i-1}))$ ,  $P \not\supseteq R^+$ .

If  $a(\mathbf{f}) < \infty$  and  $f_1, \dots, f_r \in R_1$ ,  $a(\mathbf{f})$  can only take a certain value depending on the number  $r$ . To see this define

$$a_i(R) = \inf\{m \in \mathbb{Z}; H_{R^+}^i(R)_n = 0 \text{ for all } n \geq m\}$$

for all  $i \geq 0$ , where  $H_{R^+}^i(R)$  denotes the  $i$ th local cohomology module of  $R$  with respect to  $R^+$ . For short, we will set  $a_i = a_i(R)$ .

PROPOSITION 2.2. *Let  $\mathbf{f}$  be a filter-regular sequence in  $R_1$ . Then*

$$a(\mathbf{f}) = \max\{a_i + i : i = 0, \dots, r - 1\}.$$

We shall need the following lemma in the proof of Proposition 2.2.

LEMMA 2.3. *Let  $G$  be a filter-regular element in  $R_1$ . Then*

$$a_{i+1} + 1 \leq a_i(R/gR) \leq \max\{a_1, a_{i+1} + 1\}$$

for all  $i \geq 0$ .

PROOF. By the definition of a filter-regular sequence,  $0 : g \subseteq \bigcup_{n=0}^{\infty} 0 : (R^+)^n$ . Therefore,  $0 : g$  is annihilated by some power of  $R^+$ . Hence  $H_{R^+}^i(0 : g) = 0$  for  $i \geq 1$ . Hence from the exact sequence

$$0 \rightarrow 0 : g \rightarrow R \rightarrow R/0 : g \rightarrow 0$$

we get  $H_{R^+}^i(R) = H_{R^+}^i(R/0 : g)$  for  $i \geq 1$ . Now, from the exact sequence

$$0 \rightarrow R/0 : g \xrightarrow{g} R \rightarrow R/gR \rightarrow 0$$

one can derive the exact sequence

$$H_{R^+}^i(R)_n \rightarrow H_{R^+}^i(R/gR)_n \rightarrow H_{R^+}^{i+1}(R)_{n-1} \rightarrow H_{R^+}^{i+1}(R)_n$$

for  $i \geq 0$  which immediately implies the statement.

PROOF OF PROPOSITION 2.2. Since

$$0 : f_1 \subseteq \bigcup_{n=0}^{\infty} 0 : (R^+)^n \quad \text{and} \quad f_1 H_{R^+}^0(R)_{a_0-1} \subseteq H_{R^+}^0(R)_{a_0} = 0,$$

$a(f_1) = a_0$ . Therefore, the case  $r = 1$  is immediate. For  $r > 1$  let  $\mathbf{f}'$  denote the sequence of the images of  $f_2, \dots, f_r$  in  $R/f_1R$ . By induction and using Lemma 2.3, we have

$$\begin{aligned} \max\{a_i + i; i = 1, \dots, r - 1\} &\leq a(\mathbf{f}') = \max\{a_i(R/f_1R) + i; i = 0, \dots, r - 2\} \\ &\leq \max\{a_i + i; i = 0, \dots, r - 1\}. \end{aligned}$$

Thus, since  $a(\mathbf{f}) = \max\{a(f_1), a(\mathbf{f}')\}$ , the statement is obvious.

**3. Reduction exponent.** Let  $\mathfrak{n}$  be the maximal ideal of  $R_0$ . By tensoring  $R$  with  $R_0[u]_{(n)}$ , where  $u$  is some indeterminate, we may always assume that  $k$  is infinite. Then there exist minimal reductions of  $R^+$  [9]. Let  $I$  be an arbitrary minimal reduction of  $R^+$ . The reduction exponent  $r(I)$  is just the least integer  $n$  such that  $R_{n+1} = I_{n+1}$ . The aim of this section is to produce some relationship between  $r(I)$  and the local cohomology modules of  $R$ .

Set  $r = \dim(R/\mathfrak{n}R)$ , the analytic spread of  $R^+$ . It is well known that every homogeneous minimal basis of  $I$  consists of  $r$  linear forms.

LEMMA 3.1. *There exist filter regular sequences of  $R$  minimally generating  $I$ .*

PROOF. This statement may be formulated for every homogeneous ideal  $J$  with the property  $\sqrt{J} = \sqrt{R^+}$ . Indeed, by induction on  $\dim_k(J/MJ)$ , we only need to choose a homogeneous element  $g \in J \setminus MJ$  such that  $g \notin P$  for all primes  $P \in \text{Ass}(R)$ ,  $P \not\supseteq R^+$ . That is always possible because  $k$  is infinite and  $J \setminus MJ$  is not contained in any  $P$  by the Nakayama lemma.

PROPOSITION 3.2.  $a_r + r \leq r(I) + 1 \leq \max\{a_i + i; i = 0, \dots, r\}$ .

PROOF. If  $r = 0$ ,  $R^+$  is nilpotent and  $r(I)$  is the least integer  $n$  such that  $R_{n+1} = 0$ . Therefore,  $r(I) + 1 = a_0$  because  $H_{R^+}^0(R) = R$ . If  $r > 1$ , choose a filter-regular sequence  $f_1, \dots, f_r$  generating  $I$ . Note that  $I/f_1R$  is a minimal reduction of  $R^+/f_1R$  with  $r(I/f_1R) = r(I)$ . Then the statement will follow from the induction hypothesis on  $R/f_1R$  by using Lemma 2.3.

At this point we remark that  $\max\{a_i + i; i = 0, \dots, r\} - 1$  is the Castelnuovo regularity of  $R$  [10]. In particular, if we define

$$a(I) := \inf\{a(\mathbf{f}); I = (\mathbf{f})\},$$

then we have the following relation.

COROLLARY 3.3.  $\max\{r(I) + 1, a(I)\} = \max\{a_i + i; i = 0, \dots, r\}$ .

PROOF. By Lemma 2.1, Proposition 2.2, and Lemma 3.1,

$$a(I) = \max\{a_i + i; i = 0, \dots, r - 1\}.$$

Hence the statement follows from Proposition 3.2.

LEMMA 3.4. *Suppose that  $I$  is minimally generated by a  $d$ -sequence  $\mathbf{f}$ . Then  $r(I) + 1 \geq a(I)$ .*

PROOF. By the definition of a  $d$ -sequence,

$$((f_1, \dots, f_{i-1}): f_i) \cap I = (f_1, \dots, f_{i-1})$$

for  $i = 1, \dots, r$  [8, 15]. If  $n \geq r(I) + 1$ , then  $R_n = I_n$  and hence

$$((f_1, \dots, f_{i-1}): f_i)_n = ((f_1, \dots, f_{i-1}): f_i)_n \cap I_n = (f_1, \dots, f_{i-1})_n.$$

In the following we call  $R$  a graded Buchsbaum ring if  $R_M$  is a Buchsbaum local ring [14].

**COROLLARY 3.5.** *Suppose that  $R$  is a graded Buchsbaum ring with  $\dim(R_0) = 0$ . Then  $r(I) + 1 = \max\{a_i + i; i = 0, \dots, r\}$  (independent of the choice of  $I$ ).*

**PROOF.** Since  $\dim(R_0) = 0$ ,  $I$  can be generated by a homogeneous system of parameters of  $R$ . By [8],  $R$  is Buchsbaum iff every system of parameters of  $R$  is a  $d$ -sequence. Hence the statement follows from Lemma 3.4.

Corollary 3.5 is a generalization of the formula  $r(I) = a_d + d - 1$  for Cohen-Macaulay rings  $R$  [5]. In general, if  $r(I)$  is independent of the choice of  $I$ ,  $r(I)$  need not be  $\max\{a_i + i; i = 0, \dots, r\} - 1$ .

**EXAMPLE.** Let  $R = k[X, Y]/(X^2, XY^2)$ . Then every minimal reduction of  $R^+$  has the reduction exponent 1, whereas  $a_0 = 3$ .

**4. Degree bound.** Let  $f_1, \dots, f_s$  be a minimal basis of the  $R_0$ -module  $R_1$ . Then one can define a natural map from the polynomial ring

$$R_0[X] := R_0[X_1, \dots, X_s]$$

to  $R$  by sending  $X_i$  to  $f_i, i = 1, \dots, s$ . Let  $P$  denote the kernel of this map. It is not hard to see that the set of integers occurring as degrees of elements of homogeneous minimal bases of  $P$  is independent of the choice of  $f_1, \dots, f_s$ . Let  $m_R$  denote the maximum of these integers.

**PROPOSITION 4.1.**  *$m_R \leq \max\{r(I) + 1, a(I)\}$  for every minimal reduction  $I$  of  $R^+$ .*

**PROOF.** Let  $Q$  denote the ideal of all elements of  $R_0[X]$  whose images in  $R$  belong to  $I$ . Let  $m$  denote the maximum degree of elements of all homogeneous minimal bases of  $Q$ . Since  $R_0[X]_n = Q_n$  for  $n \geq r(I) + 1, m \leq r(I) + 1$ . Hence we may assume that  $m_R > m$ . Choose  $r$  linear forms  $F_1, \dots, F_r$  such that the sequence  $\mathbf{f}$  of their images in  $R$  forms a basis of  $I$  with  $a(\mathbf{f}) = a(1)$ . Let  $F \in P$  be an arbitrary form of degree  $m_R$ . Since  $F_1, \dots, F_r$  can be extended to a minimal basis of  $Q$  by forms of  $P, F = G + \sum_{i=1}^r F_i G_i$  for some forms  $G$  and  $G_i$  such that  $G$  belongs to the ideal  $Q'$  generated by forms of  $P$  of degree  $< m_R$ . If  $m_R > a(I),$

$$G_r \in ((P, F_1, \dots, F_{r-1}) : F_r)_{m_R-1} = (P, F_1, \dots, F_{r-1})_{m_R-1}.$$

Hence we can omit the term  $G_r F_r$  in the above presentation of  $F$ . Proceeding like that, we can successively omit  $G_{r-1} F_{r-1}, \dots, G_1 F_1$  and obtain  $F \in Q',$  a contradiction.

Combining Proposition 4.1 with Corollary 3.3, we get

$$m_R \leq \max\{a_i + i; i = 0, \dots, r\}$$

which generalizes a result of Ooishi [10, Proposition 20], and Eisenbud and Goto [4, Theorem 1.2]. In particular, by Corollary 3.5, we have the following consequence.

**COROLLARY 4.2.** *Let  $R$  be a graded Buchsbaum ring with  $\dim(R_0) = 0$ . Then  $m_R \leq r(I) + 1$  for any minimal reduction  $I$  of  $R^+$ .*

**PROOF OF THEOREM 1.1.** By Corollary 4.2, it suffices to show that  $r(I) \leq e(R)$  (resp.  $e(R) - 1$ ). Since  $\dim(R_0) = 0, I$  is a minimal reduction of  $M$ . Hence  $e(R)$  is equal to the multiplicity of  $R$  with respect to  $I$ . Note that  $I$  is generated by a homogeneous system of parameters  $\mathbf{f} = f_1, \dots, f_r$  of  $R$ . If  $R$  is Cohen-Macaulay,  $r(I) \leq l(R/I) - 1 = e(R) - 1$ . If  $R$  is Buchsbaum and  $r = 1,$  consider the ring

$R' = R/H_m^0(R)$ . Then  $r(IR') \leq e(R') - 1$  because  $R'$  is a Cohen-Macaulay ring. Since  $R^+H_M^0(R) = 0$  [14], it is easy to check that  $r(IR') \geq r(I) - 1$  and  $e(R') = e(R)$ . Hence  $r(I) \leq e(R)$ . If  $r > 1$ , consider the ring  $R/f_1R$ . Then  $r(I/f_1R) = r(I)$  and by the definition of Buchsbaum rings,  $e(R/f_1R) = e(R)$ . Hence  $r(I) \leq e(R)$  by the induction hypothesis of  $R/f_1R$ .

It is worth noticing that the bound  $r(I) + 1$  of Corollary 4.2 is much better than the bound  $e(R) + 1$  of Theorem 1.1.

**EXAMPLE.** Let  $R$  be the homogeneous coordinate ring of a projective curve given parametrically by a set  $S$  of monomials of degree  $e$  in two indeterminates  $t, s$  such that  $r^e, t^{e-1}s, ts^{e-1}, s^e \in S$ . Consider  $R$  as the subring of  $k[t, s]$  generated by the monomials of  $S$ . Put  $I = (t^e, s^e)$ . Then  $r(I) \leq 2$  if  $R$  is a Buchsbaum ring [16, Theorem 4.1], whereas  $e(R) = e$  may be arbitrarily large. A famous example is the twisted cubic curve with the parameters  $t^4, t^3s, ts^3, s^4$  which is known to be arithmetically Buchsbaum.

If  $R_0 = k$ , Ooishi [10, Theorems 19 and 20] has given the bound  $m_R \leq e(R) + r - \dim(R_1) + i(R) + 1$ , where  $i(R) := \sum_{i=0}^{r-1} \binom{r-1}{i} H_M^i(R)$ . Except in the case where  $R$  is a Cohen-Macaulay ring, this bound is not better than the bound  $e(R) + 1$  of Theorem 1.1.

**EXAMPLE.** Let  $R$  be as in the above example with  $S = \{t^{3p-i+1}s^i; i = 0, 1, 3p, 3p + 1 \text{ and } i = 3j - 1 \text{ with } j = 1, \dots, p\}$ , where  $p$  is some positive integer. By [16, Theorem 4.1], it is easy to check that  $R$  is a Buchsbaum ring with  $i(R) = 2p - 2$ . Therefore  $e(R) + 1 = 3p + 2 < 4p - 2 = e(R) + r - \dim(R_1) + i(R) + 1$  if  $p > 4$ .

**5. The local case.** Let  $\mathfrak{a}$  be an arbitrary ideal of the local ring  $(A, \mathfrak{m})$ . Set  $r = \dim(G_{\mathfrak{a}}(A)/\mathfrak{m}G_{\mathfrak{a}}(A))$ , the analytic spread of  $\mathfrak{a}$ . Then  $\mathfrak{a}$  is called equimultiple if  $\text{ht}(\mathfrak{a}) = r$  [6]. This notion has some relation to Zariski's equimultiplicity and is satisfied if  $A$  is normal flat along  $\mathfrak{a}$  (an important notion in Hironaka's resolution of singularities). The class of equimultiple ideals is rather large. It contains e.g. all  $\mathfrak{m}$ -primary ideals.

**PROPOSITION 5.1.** *Let  $A$  be a Buchsbaum ring and  $\mathfrak{a}$  an equimultiple ideal. Let  $x_1, \dots, x_r$  be elements of  $\mathfrak{a}$  such that  $\mathfrak{b} = (x_1, \dots, x_r)$  is a minimal reduction of  $\mathfrak{a}$  and*

$$(x_1, \dots, x_i) \cap \mathfrak{a}^{n+1} = (x_1, \dots, x_i)\mathfrak{a}^n$$

for all  $n \geq 0, i = 1, \dots, r - 1$ . Let  $r(\mathfrak{b})$  denote the reduction exponent of  $\mathfrak{b}$ . Set  $R = G_{\mathfrak{a}}(A)$ . Then

- (i)  $r(\mathfrak{b}) = \max\{a_i + i; i = 0, \dots, r\} - 1$ ,
- (ii)  $m_R \leq r(\mathfrak{b}) + 1$ .

**PROOF.** Let  $f_1, \dots, f_r$  be the initial forms of  $x_1, \dots, x_r$  in  $R$ . Then  $I = (f_1, \dots, f_r)$  is a minimal reduction of  $R^+$  with  $r(I) = r(\mathfrak{b})$ . By Corollary 3.3 and Proposition 4.1, it suffices to show that  $r(I) + 1 \geq a(\mathfrak{f})$ , i.e.  $((f_1, \dots, f_{i-1}): f_i)_n = (f_1, \dots, f_{i-1})_n$  for  $n \geq r(I) + 1, i = 1, \dots, r$ . Translating this condition in terms of  $x_1, \dots, x_r$ , we have to show that

$$[((x_1, \dots, x_{i-1})\mathfrak{a}^n + \mathfrak{a}^{n+2}): x_i] \cap \mathfrak{a}^n = (x_1, \dots, x_{i-1})\mathfrak{a}^{n-1} + \mathfrak{a}^{n+1}.$$

Since the left side obviously contains the right side, it is sufficient to prove the converse inclusion. Let  $x$  be an arbitrary element of the left side. Then  $x_i x = y$  modulo  $(x_1, \dots, x_{i-1})\mathfrak{a}^n$  for some element  $y \in (x_1, \dots, x_i) \cap \mathfrak{a}^{n+2} = (x_1, \dots, x_{i-1})\mathfrak{a}^{n+1}$ .

Thus, there is an element  $z \in \mathfrak{a}^{n+1}$  such that  $x_i(x - z) \in (x_1, \dots, x_{i-1})$ . Since  $\text{ht}(\mathfrak{b}) = \text{ht}(\mathfrak{a}) = r$ ,  $x_1, \dots, x_r$  belong to a system of parameters of  $A$ . Therefore,  $x_1, \dots, x_r$  is a  $d$ -sequence [8]. Hence  $((x_1, \dots, x_{i-1}) : x_i) \cap \mathfrak{b} = (x_1, \dots, x_{i-1})$ . Note that  $\mathfrak{a}^n = \mathfrak{b}\mathfrak{a}^{n-1} \subseteq \mathfrak{b}$ . Then

$$\begin{aligned} x - z &\in ((x_1, \dots, x_{i-1}) : x_i) \cap \mathfrak{a}^n = (x_1, \dots, x_{i-1}) \cap \mathfrak{a}^n \\ &= (x_1, \dots, x_{i-1})\mathfrak{a}^{n-1}. \end{aligned}$$

Hence we can conclude that  $x \in (x_1, \dots, x_{i-1})\mathfrak{a}^{n-1} + \mathfrak{a}^{n+1}$ .

Now we will apply Proposition 5.1 to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Without restriction, we may assume that  $k = A/\mathfrak{m}$  is infinite. Since  $\mathfrak{a}$  is an  $\mathfrak{m}$ -primary ideal,  $\mathfrak{b}$  is generated by a system of parameters  $x_1, \dots, x_d$  of  $A$ . Set  $R = G_{\mathfrak{a}}(A)$ . By Lemma 3.1, we may assume that the initial forms  $f_1, \dots, f_d$  of  $x_1, \dots, x_d$  in  $R$  form a filter-regular sequence. Since  $\text{depth}(R) \geq d - 1$ ,  $f_1, \dots, f_{d-1}$  is a regular sequence. By [17], this implies the relation

$$(x_1, \dots, x_{i-1}) \cap \mathfrak{a}^{n+1} = (x_1, \dots, x_{i-1})\mathfrak{a}^n$$

for all  $n \geq 0, i = 1, \dots, d - 1$ . Hence (i) and (ii) follow from Lemma 5.1. For (iii) we first note that the case  $d = 1$  is already known [1, Propositions 2.2 and 2.3]. If  $d > 1$ ,  $R/f_1R = g_{\mathfrak{a}/x_1A}(A/x_1A)$  because  $f_1$  is a non-zero-divisor of  $R$ . Thus,  $\text{depth}(G_{\mathfrak{a}/x_1A}(A/x_1A)) \geq d - 2$  and  $r(\mathfrak{b}/x_1A) = r(I/f_1R) = r(I) = r(\mathfrak{b})$ , where  $I = (f_1, \dots, f_d)$ . By the induction hypothesis,

$$r(\mathfrak{b}) \leq e(A/x_1A)t^{d-2} \quad (\text{resp. } e(A/x_1A)t^{d-2} - 1).$$

On the other hand, like in [11, Lemma 1.1], we can choose  $x_1$  so that the initial form  $g$  of  $x_1$  in  $G_{\mathfrak{m}}(A)$  is filter-regular. From this it follows that  $(0 : g)_n = 0$  for  $n$  sufficiently large. Let  $s$  be the degree of  $g$  in  $G_{\mathfrak{m}}(A)$ . Then  $(\mathfrak{m}^{n+s+1} : x_1) \cap \mathfrak{m}^n = \mathfrak{m}^{n+1}$  for  $n$  sufficiently large, too. Therefore,  $x_1$  is a superficial element of order  $s$  for  $\mathfrak{m}$  and we get  $e(A/x_1A) = e(A)s$  [19, Chapter VIII, §8]. Since  $x_1 \notin \mathfrak{m}\mathfrak{a} \subseteq \mathfrak{m}^{t+1}$ ,  $s \leq t$ . Hence we can conclude that

$$r(\mathfrak{b}) \leq e(A)t^{d-1} \quad (\text{resp. } e(A)t^{d-1} - 1).$$

REMARK 5.2. In [2, Theorem 2], Cho claimed that if  $\mathfrak{a}$  is an  $\mathfrak{m}$ -primary ideal of a Cohen-Macaulay ring  $A$  such that  $R = G_{\mathfrak{a}}(A)$  is free over  $A/\mathfrak{a}$  and  $\text{depth}(R) \geq d - 1$ , then  $m_R \leq e(A)$ . But his proof for the case  $d > 1$  contains the wrong argument that there is an element  $x_1 \in \mathfrak{a}$  such that  $e(A/x_1A) = e(A)$ . For example, that is impossible if  $\mathfrak{a} \subseteq \mathfrak{m}^2$ . However, the author of the present paper could neither give another proof nor find a counterexample to this claim. The condition  $G_{\mathfrak{a}}(A)$  being free over  $A/\mathfrak{a}$  seems to be very strong. For example, if  $A$  is regular, this condition implies that  $\mathfrak{a}$  is generated by a regular sequence [18].

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