SPECTRAL SYNTHESIS ON THE ALGEBRA OF ABSOLUTELY
CONVERGENT LAGUERRE POLYNOMIAL SERIES
YÛICHI KANJIN

Abstract. Askey and Gasper [1] constructed the algebra with convolution
structure for Laguerre polynomials. We will answer the question of spectral
synthesis of the one point on this algebra.

1. Introduction. Let \( L_n^\alpha(x) \) be the Laguerre polynomial given by
\[
L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \left( \frac{d}{dx} \right)^n \left[ e^{-x} x^{n+\alpha} \right],
\]
and denote by \( R_n^\alpha(x) \) the normalized Laguerre polynomial so that
\[
R_n^\alpha(x) = L_n^\alpha(x) / L_n^\alpha(0),
\]
where \( \alpha > -1 \) and \( n \) is a nonnegative integer.

Let \( \alpha \geq -1/2 \) and \( \tau \geq 2 \) or let \( \alpha > \alpha_0 = (-5 + (17)^{1/2})/2 \) and \( \tau \geq 1 \). Let \( A(\alpha, \tau) \)
be the space
\[
\left\{ f(x) \text{ on } [0, \infty); f(x) = \sum_{n=0}^{\infty} a_n R_n^\alpha(x) e^{-\tau x}, \sum_{n=0}^{\infty} |a_n| < \infty \right\},
\]
and introduce a norm to \( A(\alpha, \tau) \) by \( \|f\| = \sum_{n=0}^{\infty} |a_n| \). Then Askey and Gasper [1]
showed that

(A) [1, §§4, 5] \( A(\alpha, \tau) \) is a Banach algebra of continuous functions on the interval
\([0, \infty)\) vanishing at infinity with the product of pointwise multiplication of functions.

Kanjin [3] studied some properties of the algebra \( A(\alpha, \tau) \) and showed that

(B) [3, THEOREM 1, COROLLARY 1] The algebra \( A(\alpha, \tau) \) is semisimple and
regular. The maximal ideal space of \( A(\alpha, \tau) \) is the interval \([0, \infty)\), and the Gelfand
transform of \( f \) in \( A(\alpha, \tau) \) is given by \( f \) itself.

(C) [3, THEOREM 2] Let \( x_0 > 0 \). If \( \alpha \geq 1/2 \) and \( \tau \geq 1 \), then the singleton \( \{x_0\} \)
is not a set of spectral synthesis for \( A(\alpha, \tau) \).

Here, a closed set \( E \) of \([0, \infty)\) is called a set of spectral synthesis for \( A(\alpha, \tau) \) if a
closed ideal \( I \) such that \( Z(I) = E \) is unique, where \( Z(I) = \{x \text{ in } [0, \infty); f(x) = 0 \}
for all \( f \) in \( I \} \).

The purpose of this paper is to solve the problem which remains unsolved in (C).

Theorem. (1) Let \( \alpha \geq -1/2 \) and \( \tau \geq 2 \) or let \( \alpha \geq \alpha_0 \) and \( \tau \geq 1 \). Then, for
every \( (\alpha, \tau) \), the singleton \( \{0\} \) is a set of spectral synthesis for \( A(\alpha, \tau) \).

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Let \( x_0 > 0 \). If \(-1/2 \leq \alpha < 1/2\) and \( \tau \geq 2 \) or if \( \alpha_0 \leq \alpha < 1/2 \) and \( \tau \geq 1 \), then the singleton \( \{x_0\} \) is a set of spectral synthesis for \( A^{(\alpha, \tau)} \).

This theorem is an immediate consequence of the following proposition which will be proved in §3.

**Proposition.** Let \( \alpha \geq -1/2 \) and \( \tau \geq 2 \) or let \( \alpha \geq \alpha_0 \) and \( \tau \geq 1 \). Let \( I \) be a closed ideal in \( A^{(\alpha, \tau)} \) such that \( Z(I) = \{x_0\}, \ x_0 \geq 0 \). If \( x_0 > 0 \), then \( I = \{f \in A^{(\alpha, \tau)}; f^{(j)}(x_0) = 0, \ j = 0, 1, \ldots, M\} \) for some \( M \leq \alpha + 1/2 \). If \( x_0 = 0 \), then \( I = \{f \in A^{(\alpha, \tau)}; f(0) = 0\} \).

Related results will be found in Cazzaniga and Meaney [2], Wolfenstetter [8], and Schwartz [6]. They are concerned with spectral synthesis on the algebra of absolutely convergent Jacobi polynomial series and on the algebra of Hankel transforms.

2. A lemma. First, we will prepare a lemma for the proof of the proposition. Let \( C_c^\infty[0, \infty) \) be the space of functions on \([0, \infty)\) which are the restrictions of infinitely differentiable functions with compact support in \((-\infty, \infty)\).

**Lemma.** Let \( \alpha \geq -1/2 \) and \( \tau \geq 2 \) or let \( \alpha \geq \alpha_0 \) and \( \tau \geq 1 \).

1. Let \( f \) be in \( C_c^\infty[0, \infty) \) and let \( q \) be the least integer greater than \( \alpha + 3/2 \). Then \( f \) is in \( A^{(\alpha, \tau)} \) and

\[
\|f\| \leq C \left( \sup_{x \geq 0} |f(x)e^{\tau x}| + Kq \sup_{x \geq 0} \left| \left( \frac{d}{dx} \right)^q f(x)e^{\tau x} \right| \right),
\]

where \( C \) is a constant depending only on \( \alpha \) and \( \tau \), and \( K \) is a number such that \( \text{supp} f \subset [0, K] \).

2. \( C_c^\infty[0, \infty) \) is dense in \( A^{(\alpha, \tau)} \).

3. Let \( f \) be in \( A^{(\alpha, \tau)} \) and let \( r \) be the greatest integer not exceeding \( \alpha + 1/2 \). Then \( f \) is \( r \)-times continuously differentiable and, for \( x \) in \((0, \infty)\) and \( j = 0, 1, \ldots, r \), there exists a constant \( B \) not depending on \( f \) such that \( |f^{(j)}(x)| \leq B \|f\| \).

**Proof.** (2) is [3, Lemma 2] and (3) is implicitly proved in the proof of [3, Theorem 2], and also, in weak form, (1) is given in [3]. Here, we will only give an outline of the proof of (1). If \( f(x) = \sum_{n=0}^\infty a_n L_n^\alpha(x)e^{-\tau x} \), then

\[
a_n = \Gamma(\alpha + 1)^{-1} \int_0^\infty f(x)e^{\tau x} L_n^\alpha(x)e^{-x}x^\alpha dx.
\]

We put \( \|f\| = \left\{ \sum_{n \leq 1/K} + \sum_{1/K < n} \right\} |a_n| = S_1 + S_2 \). For \( S_1 \), we have

\[
S_1 \leq \frac{1}{\Gamma(\alpha + 1)} \sum_{n \leq 1/K} \int_0^K |f(x)e^{\tau x}| |L_n^\alpha(x)e^{-x/2}x^\alpha e^{-x/2} dx|
\]

and, by the inequality \( |L_n^\alpha(x)e^{-x/2}x^\alpha| \leq C \) for \( 0 \leq x \leq 1/n \) (cf. [7, 8.22]), we have \( S_1 \leq C \text{sup}_{0 \leq x} |f(x)e^{\tau x}| \). Here and below, the letter \( C \) means positive constants depending only on \( \alpha \) and \( \tau \), and it may vary from inequality to inequality. From integration by parts, it follows that

\[
a_n = \frac{(n-q)!(-1)^q}{\Gamma(\alpha + 1)n!} \int_0^\infty \left\{ \left( \frac{d}{dx} \right)^q f(x)e^{\tau x} \right\} L_n^{\alpha+q}(x)e^{-x}x^{\alpha+q} dx.
\]
We have
\[ S_2 \leq \sup_{0 \leq x} \left| \left( \frac{d}{dx} \right)^q f(x) e^{rx} \right| \sum_{1/K < n} n^{-q} \int_0^K |L_{n-q}^\alpha(x)| e^{-x \alpha + q} \, dx \]
\[ \leq C \sup_{0 \leq x} \left| \left( \frac{d}{dx} \right)^q f(x) e^{rx} \right| \left\{ \sum_{1/K < n} n^{-q} \int_0^{1/n} + \sum_{1/K < n} n^{-q} \int_0^{1/n} \right\} \]
\[ \leq C \sup_{0 \leq x} \left| \left( \frac{d}{dx} \right)^q f(x) e^{rx} \right| \{I_1 + I_2\}, \text{ say.} \]

Then we have \( I_1 \leq CK^q \) and, by the inequality
\[ |L_n^\alpha(x)| \leq Ce^{-x/2}x^{-\alpha/2 - 1/4}n^{\alpha/2 - 1/4} \]
(cf. [7, 2.22]), we have \( I_2 \leq CK^q \). Q.E.D.

3. Proof of the proposition. Let \( L_I \) be the space of continuous linear functionals \( \phi \) on \( A^{(\alpha, r)} \) such that \( \phi(f) = 0 \) for all \( f \) in \( I \). We will show that, if \( \phi \) is in \( L_I \), then \( \phi \) is of the form

\[
\phi(f) = \begin{cases} 
\sum_{j=0}^p a_j \delta_{z_0}^{(j)}(f), & p \leq \alpha + 1/2 \ (x_0 > 0), \\
\alpha_0 \delta_{z_0}(f), & (x_0 = 0)
\end{cases}
\]

for \( f \) in \( A^{(\alpha, r)} \), where \( \delta_{z_0}^{(j)} \) is the functional such that \( \delta_{z_0}^{(j)}(f) = f^{(j)}(z_0) \) for \( f \) in \( A^{(\alpha, r)} \). Then the proposition is proved as follows. Let \( p(\phi) = \max\{j; a_j \neq 0\} \) for \( \phi \) in \( L_I \), and \( M = \max\{p(\phi); \phi \in L_I\} \). By (*), we have that \( M = 0 \) for \( x_0 = 0 \) and \( 0 \leq M \leq \alpha + 1/2 \) for \( x_0 > 0 \). Let \( \phi_0 \) be a functional in \( L_I \) such that \( M = p(\phi_0) \).

From (1) in the lemma it follows that there exist functions \( h_m \) in \( A^{(\alpha, r)} \) such that \( h_m^{(k)}(x_0) = \delta_{mk} \), \( k, m = 0, 1, \ldots, M \), where \( \delta_{mk} \) is Kronecker's symbol. For every \( f \) in \( I \), we have

\[ 0 = \phi_0(fh_m) = \sum_{k=0}^M \left\{ \sum_{j=k}^M jC_k a_j f^{(j-k)}(x_0) \right\} h_m^{(k)}(x_0) \]
\[ = \sum_{j=m}^M jC_m a_j f^{(j-m)}(x_0), \quad m = 0, 1, \ldots, M. \]

Thus \( f^{(j)}(x_0) = 0 \) for \( j = 0, 1, \ldots, M \). This implies that \( I = \{ f \in A^{(\alpha, r)}; f^{(j)}(x_0) = 0, j = 0, 1, \ldots, M \} \) since \( I \) is the space of \( f \) in \( A^{(\alpha, r)} \) such that \( \phi(f) = 0 \) for all \( \phi \) in \( L_I \).

Now we will prove (*). Let \( D(-\infty, \infty) \) be the test function space on \( (-\infty, \infty) \) with usual topology. For \( f \) in \( D(-\infty, \infty) \), we put \( f_P(x) = f(x), x \geq 0, \) and \( f_N(x) = f(-x), x \geq 0 \). Then, by (1) in the lemma, we have that \( f_P \) and \( f_N \) are in \( A^{(\alpha, r)} \). Let \( \phi \) be in \( L_I \). We define \( \Phi_+(f) = \phi(f_P) + \phi(f_N) \) and \( \Phi_-(f) = \phi(f_P) - \phi(f_N) \) for \( f \) in \( D(-\infty, \infty) \). By (1) again, we have

\[ |\Phi_\pm(f)| \leq \|\phi\|(\|f_P\| + \|f_N\|) \]
\[ \leq C\|\phi\|e^{rK} \left( \sup_{-\infty < x < \infty} |f(x)| + K^q \sum_{j=1}^q \sup_{-\infty < x < \infty} |f^{(j)}(x)| \right), \]
where $K$ is a number such that $\text{supp} f \subset [-K, K]$, and $q$ is the least integer greater than $\alpha + 3/2$. Thus $\Phi_{\pm}$ are continuous linear functionals on $D(-\infty, \infty)$ with order not exceeding $q$. Since $A^{(\alpha, \tau)}$ is regular, the ideal $I$ contains the ideal of functions in $A^{(\alpha, \tau)}$ which vanish on a neighborhood of $x_0$ (cf. [4, 5.7]). This implies that the supports of $\Phi_{\pm}$ are the singleton $\{x_0\}$. Thus $\Phi_{\pm}$ have the forms

$$\Phi_{+} = \sum_{j=0}^{q} a_{j}^{+} \delta_{x_0}^{(j)}, \quad \Phi_{-} = \sum_{j=0}^{q} a_{j}^{-} \delta_{x_0}^{(j)},$$

where the $a_{j}^{\pm}$ are constants (cf. [5, 6.25]).

We will show that $a_{j}^{\pm} = 0$ for $j > \alpha + 1/2$ if $x_0 > 0$. Let $u(x)$ be a function in $D(-\infty, \infty)$ such that $u(x) = 1$ on a neighborhood of $x_0$ and $\text{supp} u \subset (0, \infty)$. Then the function $u(x)e^{-\tau x R_{n}^{\alpha}(x)}$ is in $D(-\infty, \infty)$, and

$$|\Phi_{\pm}(ue^{-\tau x R_{n}^{\alpha}})| \leq \|\phi(ue^{-\tau x R_{n}^{\alpha}})\| \leq \|\phi\| \|u\|$$

since $\|(e^{-\tau x R_{n}^{\alpha}})_{P}\| = 1$. In particular, $\Phi_{\pm}(ue^{-\tau x R_{n}^{\alpha}}) = O(1)$ ($n \to \infty$). On the other hand, by the formula $(d/dx) L_{n}^{\alpha}(x) = -L_{n}^{\alpha+1}(x)$ (cf. [7, (5.1.14)]) and the asymptotic formula

$$L_{n}^{\alpha}(x) = \pi^{-1/2} e^{x/2} \alpha^{-1/2} \cos\left(2n(x)^{1/2} - \alpha \pi / 2 - \pi / 4\right)$$

and

$$L_{n}^{\alpha+1}(x) = \pi^{-1/2} e^{x/2} \alpha^{-1/2} \cos\left(2n(x)^{1/2} - \alpha \pi / 2 - \pi / 4\right)$$

(cf. [7, (8.22.1)]), we have

$$\delta_{x_0}^{(j)}(ue^{-\tau x R_{n}^{\alpha}}) = O(n^{-(\alpha-j)/2-1/4})$$

and

$$\limsup_{n \to \infty} |\delta_{x_0}^{(j)}(ue^{-\tau x R_{n}^{\alpha}})| n^{(\alpha-j)/2+1/4} > 0$$

for $j = 0, 1, \ldots, q$. This implies that

$$\limsup_{n \to \infty} |\Phi_{\pm}(ue^{-\tau x R_{n}^{\alpha}})| = \infty$$

if $a_{j}^{\pm} \neq 0$ for some $j > \alpha + 1/2$. Thus we have $a_{j}^{\pm} = 0$ for $j > \alpha + 1/2$.

Next we will show that $a_{j}^{\pm} = 0$ for $j > 0$ if $x_0 = 0$. Let $u_1(x)$ be an even function in $D(-\infty, \infty)$ such that $u_1(x) = 1$ for $x$ in $[-1/2, 1/2]$ and $u_1(x) = 0$ for $x$ not in $(-1, 1)$. Put $u_n(x) = u_1(nx)$, $n = 2, 3, \ldots$, and consider the function $u_n(x)e^{-\tau x R_{n}^{\alpha}(x)}$, $-\infty < x < \infty$. Then we have

$$|\Phi_{\pm}(u_n e^{-\tau x R_{n}^{\alpha}})| \leq \|\phi((u_n e^{-\tau x R_{n}^{\alpha}})_{N})\| + \|\phi((u_n e^{-\tau x R_{n}^{\alpha}})_{P})\|.$$
We have that
\[
\left( \frac{d}{dx} \right)^j u_n(x) R_n^\alpha(x) = \sum_{k=0}^{j} j! C_k n^{j-k} u_1^{(j-k)}(nx) \times \frac{(-1)^k \Gamma(n+1) \Gamma(\alpha+k+1)}{(\alpha+1)^k \Gamma(n+\alpha+1)} f_{n-k}^\alpha(x), \quad j = 0, 1, 2, \ldots.
\]

By Perron's formula in the complex domain (see [7, (8.22.3)]), we have
\[
\sup_{-1/n \leq x \leq 0} \left| \left( \frac{d}{dx} \right)^j u_n(x) R_n^\alpha(x) \right| = O(n^j) \quad (n \to \infty), \quad j = 0, 1, 2, \ldots,
\]
and thus we have \( \| (u_n e^{-\tau x} R_n^\alpha) \| = O(1) \) \( (n \to \infty) \). Since
\[
|\phi((u_n e^{-\tau x} R_n^\alpha))| \leq \| \phi \| \| (u_n e^{-\tau x} R_n^\alpha) \|,
\]
we have the claim \( \phi((u_n e^{-\tau x} R_n^\alpha)) = O(1) \) \( (n \to \infty) \). Therefore, we have
\[
\Phi_\pm(u_n e^{-\tau x} R_n^\alpha) = O(1) \quad (n \to \infty).
\]

On the other hand, we have
\[
\Phi_\pm(u_n e^{-\tau x} R_n^\alpha) = \sum_{j=0}^{q} a_j^\pm n^{-j} \sum_{k=0}^{j} j! C_k n^{j-k} \frac{(n-1) \cdots (n-k+1)}{(\alpha+1)^k}.
\]

This implies that, if \( a_j^\pm \neq 0 \) for some \( j > 0 \), then
\[
\lim_{n \to \infty} |\Phi_\pm(u_n e^{-\tau x} R_n^\alpha)| = \infty.
\]
Thus we have that \( a_j^\pm = 0 \) for \( j > 0 \), and therefore we have that \( \phi(f_P) \) is of the form
\[
\phi(f_P) = (\Phi_+(f) + \Phi_-(f))/2
\]
\[
= \begin{cases} 
\sum_{j=0}^{\alpha} a_j \delta_{x_0}(f), & p \leq \alpha + 1/2 \ (x_0 > 0), \\
a_0 \delta_{x_0}(f), & (x_0 = 0)
\end{cases}
\]
for \( f \) in \( D(-\infty, \infty) \). From (2) and (3) in the lemma and \( |f(0)| \leq \|f\| \), it follows that \( \phi \) is of the form \((\ast)\). Q.E.D.

References


DEPARTMENT OF MATHEMATICS, COLLEGE OF LIBERAL ARTS, KANAZAWA UNIVERSITY, KANAZAWA 920, JAPAN