

SHORTER NOTES

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FOURIER SERIES WITH POSITIVE COEFFICIENTS

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ABSTRACT. Extending a result of N. Wiener, it is shown that functions on the circle with positive Fourier coefficients that are p th power integrable near 0, $1 < p \leq 2$, have Fourier coefficients in $l^{p'}$.

The following result was proved (but never published) by Norbert Wiener in the early 1950's. (See [1, pp. 242, 250] and [3].)

WIENER'S THEOREM. *If $\sum c_n e^{int}$ is the Fourier series of a function $f \in L^1(-\pi, \pi)$ with $c_n \geq 0$ for all n , and f restricted to a neighborhood $(-\delta, \delta)$ of the origin belong to $L^2(-\delta, \delta)$, then f belongs to $L^2(-\pi, \pi)$.*

A question which immediately arises in connection with this result is the following: does the theorem remain true if one replaces $L^2(-\delta, \delta)$ and $L^2(-\pi, \pi)$ in its statement respectively by $L^p(-\delta, \delta)$ and $L^p(-\pi, \pi)$, with $1 < p \leq \infty$? In 1969 Stephen Wainger showed, by ingenious counterexamples, that the answer is negative for $1 < p < 2$ [4]. If p is an even integer or ∞ it is very easy to see that the answer is "yes." For every other exponent between 2 and ∞ it is "no," as was shown in 1975 by Harold S. Shapiro [3]. These negative results have been extended to compact abelian groups [2]. However, the conclusion of Wiener's theorem can be stated equivalently as "then $\sum c_n^2 < \infty$." This suggested the following theorem.

THEOREM. *If $\sum c_n e^{int}$ is the Fourier series of a function $f \in L^1(-\pi, \pi)$ with $c_n \geq 0$ for all n , and f restricted to a neighborhood $(-\delta, \delta)$ of the origin belongs to $L^p(-\delta, \delta)$ with $1 < p < 2$, then $\sum c_n^{p'} < \infty$, where $p' = p/(p-1)$.*

PROOF. (See [3, p. 12].) Let $h(t)$ be the 2π -periodic function which for $|t| \leq \pi$ is defined by

$$h(t) = \begin{cases} 1 - |t|/\delta, & |t| \leq \delta, \\ 0, & \delta \leq |t| \leq \pi. \end{cases}$$

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Then $|hf| \leq \chi_{[-\delta, \delta]}|f| \in L^p(-\pi, \pi)$ and $\sum |(hf)^\wedge(n)|^{p'}$ is finite by the Hausdorff-Young inequality. (See (*) below.) We have

$$(h \cdot f)^\wedge(n) = \sum_{k+l=n} \hat{h}(k)c_l$$

where $c_l \geq 0$ for all l by hypothesis and $\hat{h}(k) \geq 0$ for all k by direct calculation. Drop all terms except the $k = 0$ one from the right side of the last equation to get

$$c_n \leq \frac{(h \cdot f)^\wedge(n)}{\hat{h}(0)} = \frac{2\pi}{\delta}(h \cdot f)^\wedge(n).$$

Take p' th powers and sum over n .

REMARKS. 1. This theorem was motivated by studying the above-mentioned counterexamples of Wainger [4].

2. It is very well known that the Hausdorff-Young theorem consists of two irreversible implication, one of which is

(*) if $\sum c_n e^{inx}$ is the Fourier series of a function $f \in L^p(-\pi, \pi)$, where $1 < p < 2$, then $\{c_n\} \in l^{p'}$. (See [5, pp. 101-103].)

Wainger's counterexamples are functions designed to satisfy the hypotheses of our theorem while violating the hypothesis of (*). Our theorem shows that they must also satisfy the conclusion of (*), and thereby gives another demonstration that the converse of (*) is false.

3. Our theorem easily extends to compact abelian groups. In the above proof simply replace $[-\pi, \pi]$ by a general compact abelian group and $[-\delta, \delta]$ by a symmetric neighborhood of the identity, note that $h = \varphi \star \tilde{\varphi}$ where $\varphi(t) = \tilde{\varphi}(-t) = 1/\sqrt{\delta}\chi_{[-\delta/2, \delta/2]}(t)$, etc.

REFERENCES

1. R. P. Boas, *Entire functions*, Academic Press, New York, 1964.
2. M. Rains, *On functions with nonnegative Fourier transforms*, Indian J. Math. **27** (1985), 41-48.
3. H. S. Shapiro, *Majorant problems for Fourier coefficients*, Quart. J. Math. Oxford (2) **26** (1975), 9-18.
4. S. Wainger, *A problem of Wiener and the failure of a principle for Fourier series with positive coefficients*, Proc. Amer. Math. Soc. **20** (1969), 16-18.
5. A. Zygmund, *Trigonometric series*, Vol. II, Cambridge Univ. Press, Cambridge, 1959.

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