PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 3, November 1987

A SIMPLE PROOF OF JACOBI'S FOUR-SQUARE THEOREM

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(Communicated by Larry J. Goldstein)

ABSTRACT. Jacobi's four-square theorem, which gives the number of representations of a positive integer as a sum of four squares, is shown to follow simply from the triple-product identity.

1. Recently [**2**], I showed how one can obtain Jacobi's two-square theorem from the triple-product identity. In this note I show that the triple-product identity also gives Jacobi's four-square theorem:

THEOREM 1. *The number $r_4(n)$ of representations of the positive integer n as a sum of four squares is given by*

$$r_4(n) = 8 \sum_{d \mid n,\, 4 \nmid d} d.$$

2. The triple product identity is

$$\prod_{n \geqslant 1} (1 + ax^{2n-1})(1 + a^{-1}x^{2n-1})(1 - x^{2n}) = \sum_{-\infty}^{\infty} a^n x^{n^2}.$$

It follows easily (loc. cit.) that

$$(1) \quad (a - a^{-1}) \prod_{n \geqslant 1} (1 - a^2x^n)(1 - a^{-2}x^n)(1 - x^n) = \sum_{-\infty}^{\infty} (-1)^n a^{2n+1} x^{(n^2+n)/2}.$$

Differentiate (1) with respect to a, put $a = 1$, divide by 2 and we obtain the identity

$$(2) \qquad \prod_{n \geqslant 1} (1 - x^n)^3 = \frac{1}{2} \sum_{-\infty}^{\infty} (-1)^n (2n + 1) x^{(n^2+n)/2}.$$

This is a celebrated identity of Jacobi [**1**, Theorem 357], and can be considered our starting point.

Squaring (2) gives

$$\prod_{n \geqslant 1} (1 - x^n)^6 = \frac{1}{4} \sum_{m,\, n = -\infty}^{\infty} (-1)^{m+n} (2m + 1)(2n + 1) x^{(m^2+n^2+m+n)/2}.$$

Received by the editors April 15, 1986 and, in revised form, July 15, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11E25, 11P05.

Now split the sum on the right into two, according as $m + n$ is even or odd, to obtain

$$\prod_{n\geqslant 1}(1-x^n)^6 = \frac{1}{4}\left\{\sum_{m\equiv n \pmod 2}(2m+1)(2n+1)x^{(m^2+n^2+m+n)/2} - \sum_{m\not\equiv n \pmod 2}(2m+1)(2n+1)x^{(m^2+n^2+m+n)/2}\right\}.$$

In the first sum, set $r = \frac{1}{2}(m+n)$, $s = \frac{1}{2}(m-n)$, and in the second set $r = \frac{1}{2}(m-n-1)$, $s = \frac{1}{2}(m+n+1)$, and, remarkably, the two sums coalesce to give

$$\prod_{n\geqslant 1}(1-x^n)^6 = \frac{1}{2}\sum_{r,\,s=-\infty}^{\infty}\left((2r+1)^2-(2s)^2\right)x^{r^2+s^2+r}.$$

Once again splitting the sum, we obtain

$$\prod_{n\geqslant 1}(1-x^n)^6 = \frac{1}{2}\left\{\sum_{s=-\infty}^{\infty}x^{s^2}\sum_{r=-\infty}^{\infty}(2r+1)^2x^{r^2+r} - \sum_{r=-\infty}^{\infty}x^{r^2+r}\sum_{s=-\infty}^{\infty}(2s)^2x^{s^2}\right\}$$

$$= \frac{1}{2}\left\{\sum_{s=-\infty}^{\infty}x^{s^2}\times\left(1+4x\frac{d}{dx}\right)\sum_{r=-\infty}^{\infty}x^{r^2+r} - \sum_{r=-\infty}^{\infty}x^{r^2+r}\times 4x\frac{d}{dx}\sum_{s=-\infty}^{\infty}x^{s^2}\right\}.$$

Making use of the triple-product identity, we obtain

$$\prod_{n\geqslant 1}(1-x^n)^6 = \frac{1}{2}\left\{\prod_{n\geqslant 1}(1+x^{2n-1})^2(1-x^{2n}) \times\left(1+4x\frac{d}{dx}\right)2\prod_{n\geqslant 1}(1+x^{2n})^2(1-x^{2n}) - 2\prod_{n\geqslant 1}(1+x^{2n})^2(1-x^{2n}) \times 4x\frac{d}{dx}\prod_{n\geqslant 1}(1+x^{2n-1})^2(1-x^{2n})\right\}.$$

Employing the product rule to evaluate the derivatives, we find

$$\prod_{n\geqslant 1}(1-x^n)^6 = \prod_{n\geqslant 1}(1+x^{2n-1})^2(1-x^{2n})(1+x^{2n})^2(1-x^{2n}) \times\left\{1+8\sum_{n\geqslant 1}\frac{2nx^{2n}}{1+x^{2n}} - 4\sum_{n\geqslant 1}\frac{2nx^{2n}}{1-x^{2n}}\right\} - \prod_{n\geqslant 1}(1+x^{2n})^2(1-x^{2n})(1+x^{2n-1})^2(1-x^{2n}) \times\left\{8\sum_{n\geqslant 1}\frac{(2n-1)x^{2n-1}}{1+x^{2n-1}} - 4\sum_{n\geqslant 1}\frac{2nx^{2n}}{1-x^{2n}}\right\},$$

or,

$$\prod_{n\geqslant 1}(1-x^n)^6 = \prod_{n\geqslant 1}(1+x^{2n-1})^2(1+x^{2n})^2(1-x^{2n})^2 \times\left\{1-8\sum_{n\geqslant 1}\left(\frac{(2n-1)x^{2n-1}}{1+x^{2n-1}} - \frac{2nx^{2n}}{1+x^{2n}}\right)\right\}.$$

Dividing both sides by

$$\prod_{n\geqslant 1}(1+x^n)^4(1-x^n)^2 = \prod_{n\geqslant 1}(1+x^n)^2(1-x^{2n})^2$$
$$= \prod_{n\geqslant 1}(1+x^{2n-1})^2(1+x^{2n})^2(1-x^{2n})^2,$$

we obtain

$$\prod_{n\geqslant 1}\left(\frac{1-x^n}{1+x^n}\right)^4 = 1 - 8\sum_{n\geqslant 1}\left(\frac{(2n-1)x^{2n-1}}{1+x^{2n-1}} - \frac{2nx^{2n}}{1+x^{2n}}\right).$$

Now, it is a simple consequence of the triple-product identity that

$$\prod_{n\geqslant 1}\left(\frac{1-x^n}{1+x^n}\right) = \sum_{-\infty}^{\infty}(-1)^n x^{n^2},$$

so we have

$$\left(\sum_{-\infty}^{\infty}(-1)^n x^{n^2}\right)^4 = 1 - 8\sum_{n\geqslant 1}\left(\frac{(2n-1)x^{2n-1}}{1+x^{2n-1}} - \frac{2nx^{2n}}{1+x^{2n}}\right).$$

Putting $-x$ for x, we obtain

$$\left(\sum_{-\infty}^{\infty}x^{n^2}\right)^4 = 1 + 8\sum_{n\geqslant 1}\left(\frac{(2n-1)x^{2n-1}}{1-x^{2n-1}} + \frac{2nx^{2n}}{1+x^{2n}}\right)$$
$$= 1 + 8\sum_{n\geqslant 1}\left(\frac{(2n-1)x^{2n-1}}{1-x^{2n-1}} + \frac{2nx^{2n}}{1-x^{2n}}\right) - 8\sum_{n\geqslant 1}\left(\frac{2nx^{2n}}{1-x^{2n}} - \frac{2nx^{2n}}{1+x^{2n}}\right)$$
$$= 1 + 8\sum_{n\geqslant 1}\frac{nx^n}{1-x^n} - 8\sum_{n\geqslant 1}\frac{4nx^{4n}}{1-x^{4n}},$$

or,

$$\left(\sum_{-\infty}^{\infty}x^{n^2}\right)^4 = 1 + 8\sum_{4\nmid n}\frac{nx^n}{1-x^n}$$

from which Theorem 1 follows directly.

References

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Clarendon Press, Oxford.

2. Michael D. Hirschhorn, *A simple proof of Jacobi's two-square theorem*, Amer. Math. Monthly **92** (1985), 579–580.

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