ABSTRACT. We prove that the $L^p$-norm with respect to the normalized Lebesgue measure on the sphere of any spherical harmonic of degree $k$ is bounded by a constant independent of the dimension times its $L^2$-norm. Several consequences are obtained from this result.

1. Introduction. In a previous paper [2] we were led to study the quotient $\|Y_k\|_p/\|Y_k\|_2$ where $Y_k$ stands for a spherical harmonic of degree $k$ in $\mathbb{R}^n$ and the $L^p$-norms are taken with respect to the normalized Lebesgue measure on the sphere $S^{n-1}$, i.e.,

$$\|Y_k\|_p^p = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |Y_k(u)|^p \, d\sigma(u)$$

($d\sigma(u)$ = Lebesgue measure on $S^{n-1}$, $|S^{n-1}|$ = measure of $S^{n-1} = 2\pi^{n/2}\Gamma(n/2)^{-1}$).

When $p > 2$, Hölder's inequality provides the trivial lower bound 1. We prove here, by using two different approaches, that we have the upper bound $(p-1)^{1/2}$, independent of $n$. The first proof is the same as in [2] with further precision; the second uses two well-known but far from trivial facts: the Bochner-Hecke formula and the Beckner-Hausdorff-Young inequality.

For $p < 2$, Hölder's inequality gives the upper bound 1 and a lower bound independent of $n$ can be found by using the preceding part and interpolation.

Similar results can be obtained for any polynomial of degree $k$ and any sphere in $\mathbb{R}^n$ using the decomposition of the polynomial restricted to the sphere as a sum of spherical harmonics. Some other consequences are also given.

We use the same notation for the spherical harmonic $Y_k$ defined on $S^{n-1}$ and the solid harmonic defined in all $\mathbb{R}^n$ which are related by $Y_k(x) = Y_k(x/|x|)|x|^k$.

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2. The main theorem and its two proofs.

THEOREM 1. Let $Y_k$ be a spherical harmonic of degree $k$ in $\mathbb{R}^n$. Then, if $p \geq 2$,

$$\|Y_k\|_p \leq (p-1)^{1/2}\|Y_k\|_2.$$

PROOF. We use induction on $k$. Let $k = 1$. By rotation it is enough to prove the theorem for $Y_1(u) = u_1$. But a simple computation gives

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} |u_1|^p \, d\sigma(u) = \frac{\Gamma(n/2)\Gamma((p+1)/2)}{\pi^{1/2}\Gamma((n+p)/2)}$$
and \(\|Y_1\|_2 = n^{-1/2}\); then it follows from the properties of the gamma function that
\[
\|Y_1\|_p/\|Y_1\|_2 \leq (p - 1)^{1/2}.
\]

Assume now the theorem for \(k - 1\) and let \(Y_k\) be a spherical harmonic of degree \(k\). We claim that it is enough to obtain
\[
\|Y_k\|_p \leq \left(\frac{p - 1}{k(kp + n - 2)}\right)^{1/2} \|\nabla Y_k\|_p \quad \text{with equality for } p = 2.
\]

In fact, the induction hypothesis and Minkowski’s inequality imply
\[
\|\nabla Y_k\|_p \leq (p - 1)^{(k-1)/2} \|\nabla Y_k\|_2
\]
and then using (1) successively with \(L^p\)- and \(L^2\)-norms, we get
\[
\|Y_k\|_p \leq \left(\frac{p - 1}{k(kp + n - 2)}\right)^{1/2} (p - 1)^{(k-1)/2} \|\nabla Y_k\|_2
\]
\[
= (p - 1)^{k/2} \left(\frac{2k + n - 2}{kp + n - 2}\right)^{1/2} \|Y_k\|_2 \leq (p - 1)^{k/2} \|Y_k\|_2.
\]

Finally we prove (1). From the homogeneity of solid harmonic \(Y_k\) and Green’s formula we have
\[
\int_{S^{n-1}} |Y_k|^p \, d\sigma = \frac{1}{k} \int_{S^{n-1}} |Y_k|^{p-1} \text{sgn} Y_k \frac{\partial Y_k}{\partial \nu} \, d\sigma
\]
\[
= \frac{1}{k} \int_{|x|<1} \nabla (|Y_k|^{p-1} \text{sgn} Y_k) \nabla Y_k \, dx
\]
\[
= \frac{p - 1}{k} \int_{|x|<1} |Y_k|^{p-2} |\nabla Y_k|^2 \, dx
\]
\[
= \frac{p - 1}{k(kp + n - 2)} \int_{S^{n-1}} |Y_k|^{p-2} |\nabla Y_k|^2 \, d\sigma.
\]
When \(p = 2\) we get the equality in (1); for \(p > 2\), just apply Hölder’s inequality with exponents \(p/(p - 2)\) and \(p/2\). \(\Box\)

The second way to prove Theorem 1 gives a somewhat more precise result:

**Theorem 1'**. Let \(Y_k\) be a spherical harmonic of degree \(k, 2 < p < \infty\) and \(1/p + 1/q = 1\). Then,
\[
\|Y_k\|_p \leq (p - 1)^{k/2} \|Y_k\|_q.
\]

**Proof.** Bochner-Hecke’s formula states that
\[
\mathcal{F}(e^{-\pi|x|^2} Y_k(x)) = i^{-k} e^{-\pi|\xi|^2} Y_k(\xi)
\]
(\(\mathcal{F}\) is the Fourier transform and \(Y_k\) is here the solid harmonic; see Stein [4, p. 71].) The Hausdorff-Young inequality can be written in the form
\[
\|\mathcal{F}(f)\|_{L^p(R^n)} \leq (q^{1/q}/p^{1/p})^{n/2} \|f\|_{L^q(R^n)}
\]
(see Beckner [1]). When this inequality is applied to (2), taking into account that
\[
\|e^{-\pi|x|^2} Y_k(x)\|_{L^p(R^n)} = \frac{\Gamma((kp + n)/2)}{\pi^{kp/2}p(kp + n)/2\Gamma(n/2)} \|Y_k\|_p
\]
we get

\[ \frac{\|Y_k\|_p}{\|Y_k\|_q} \leq \left( \frac{p}{q} \right)^{k/2} \frac{\Gamma((kq + n)/2)^{1/q} \Gamma(n/2)^{1/p}}{\Gamma((kp + n)/2)^{1/p} \Gamma(n/2)^{1/q}}. \]

Since \( \log \Gamma \) is a convex function in \((0, \infty)\), the last factor is \( \leq 1 \) and the theorem is proved. \( \square \)

The function \( Y_k(x) = (x_1 + ix_2)^k \) is harmonic and homogeneous of degree \( k \). If we compute the \( L^p \)-norm of its restriction to \( S^{n-1} \) we see that the preceding results are sharp in the following sense: for constants independent of \( n \), no exponent less than \( k/2 \) can appear in the right-hand side of Theorems 1 and 1'; in fact

\[ \sup_n \frac{\|Y_k\|_p}{\|Y_k\|_2} \leq c(k)p^{k/2}. \]

3. Some consequences. (a) Theorem 1 and the method of rotations give the following: Let \( \{Y_j\} \) be a basis of the linear space of spherical harmonics of degree \( k \) in \( \mathbb{R}^n \) and \( d(k, n) \) its dimension. If we normalize the \( Y_j \) in such a way that \( \|Y_j\|_2 = d(k, n)^{-1/2} \) and define the operators \( (R_j f)(\xi) = Y_j(\xi/|\xi|) f(\xi) \), there then exists \( C_{p,k} \) independent of \( n \) such that

\[ \left\| \left( \sum_{j=1}^{d(k, n)} |R_j f|^2 \right)^{1/2} \right\|_p \leq C_{p,k} \|f\|_p, \quad 1 < p < \infty, \]

with \( C_{p,k} = O(p^{1+k/2}) \), \( p \to \infty \), and \( = O((p - 1)^{-1}) \), \( p \to 1 \). This was our motivation for Theorem 1 and can be seen in [2].

(b) For \( p < 2 \), a lower bound for \( \|Y_k\|_p/\|Y_k\|_2 \) can be obtained from Theorem 1, namely

**COROLLARY 2.** If \( 0 < p < 2 \),

\[ \|Y_k\|_2 \leq e^{k((2/p) - 1)} \|Y_k\|_p. \]

**PROOF.** Let \( s > 2 \). By interpolation and Theorem 1

\[ \|Y_k\|_2 \leq \|Y_k\|^\theta_p \|Y_k\|^{1-\theta}_2 \leq (s - 1)^{(1-\theta)k/2} \|Y_k\|_p \|Y_k\|^{1-\theta}_2 \]

with \( 1/2 = \theta/p + (1 - \theta)/s \). Then,

\[ \|Y_k\|_2 \leq (s - 1)^{(1-\theta)/\theta} \|Y_k\|_p \]

and the corollary follows from

\[ \inf_{s > 2} (s - 1)^{(1-\theta)/\theta} = \lim_{s \to 2} (s - 1)^{(1-\theta)/\theta} = e^{2(2/p - 1)}. \]  \( \square \)

(c) Theorem 1 and Corollary 2 have the following similar versions in the case of arbitrary polynomials.

**COROLLARY 3.** Let \( P_k \) be any polynomial of degree \( k \) and \( S \) any sphere in \( \mathbb{R}^n \). Then, if \( 2 < p < \infty \),

\[ \left( \frac{1}{|S|} \int_S |P_k|^p \, d\sigma \right)^{1/p} \leq p^{k/2} \left( \frac{1}{|S|} \int_S |P_k|^2 \, d\sigma \right)^{1/2} \]
and if $0 < p < 2$,
\[
\left( \frac{1}{|S|} \int_S |P_k|^2 \, d\sigma \right)^{1/2} \leq 4^{k(2/p-1)} \left( \frac{1}{|S|} \int_S |P_k|^p \, d\sigma \right)^{1/p}.
\]

**PROOF.** Since translation and dilation change a polynomial into another of the same degree, it will be enough to prove the result for $S = S^{n-1}$. But on $S^{n-1}$, $P_k = \sum_{j=0}^k Y_j$ where $Y_j$ is a spherical harmonic of degree $j$. By Theorem 1,
\[
\|P_k\| \leq \sum_{j=0}^k \|Y_j\| \leq \sum_{j=0}^k (p-1)^{j/2} \|Y_j\|_2
\]
and the first part of the corollary is a consequence of the orthogonality of the $Y_j$ after applying Cauchy-Schwarz inequality.

The second part follows from the first as in the proof of Corollary 2. \( \square \)

(d) The size of the constants in Theorem 1 and Corollary 3 makes it possible to give an estimate of exponential type with constant independent of $n$.

**COROLLARY 4.** Let $P_k$ be a polynomial of degree $k$ in $\mathbb{R}^n$. Then, for any sphere $S$ in $\mathbb{R}^n$
\[
\frac{1}{|S|} \int_S \exp \left( \frac{P_k(u)}{\|P_k\|_2} \right)^\lambda \, d\sigma(u) \leq C(k, \lambda)
\]
with a constant independent of $n$ if $\lambda < 2/k$ (also for $\lambda = 2/k$ if $k \geq 6$).

(e) The following application is based on a result in probability theory.

Let $Y^{(n)} = (Y_1, \ldots, Y_n, 0, 0, \ldots)$, $n = 1, 2, \ldots$, be random variables such that $(Y_1, \ldots, Y_n)$ are uniformly distributed in a sphere of $\mathbb{R}^n$ of radius $r_n = (n/2\pi)^{1/2}$ and let $X = (X_1, \ldots, X_m, \ldots)$ be a random variable with $X_1, \ldots, X_m, \ldots$ independent and having $\exp(-\pi|x|^2)$ as distribution function. Then, the sequence $Y^{(n)}$ converges to $X$ in law.

This result is known and can be easily proved. If $E_n(f)$ stands for the expectation of $f$ with respect to $Y^{(n)}$ and $E(f)$ is the expectation with respect to $X$, we have
\[
\lim_{n \to \infty} E_n(f) = E(f).
\]
Taking now as $f$ the $p$th power of a polynomial of degree $k$ and using Corollary 3 we have

**COROLLARY 5.** Let $X = (X_1, \ldots, X_n, \ldots)$ be a random variable where the $X_i$ are independent and have $\exp(-\pi|x|^2)$ as distribution function. If $P(X)$ is a polynomial of degree $k$ in the variables $X_1, \ldots, X_n, \ldots$ the following reverse Hölder inequalities hold.
\[
E(|P|^p)^{1/p} \leq p^{k/2} E(|P|^2)^{1/2}, \quad 2 < p < \infty,
\]
\[
E(|P|^2)^{1/2} \leq 4^{k(2/p-1)} E(|P|^p)^{1/p}, \quad 0 < p < 2.
\]

(f) Upper bounds for the quotient $\|Y_k\|_p/\|Y_k\|_2$, $2 < p < \infty$, are interesting also in the study of Bochner-Riesz operators on the sphere. In that case the dimension $n$ of the underlying space is kept fixed and the interest is in the behaviour of the quotient for $k \to \infty$. Sharp bounds have been obtained by C. Sogge [3].

\[
\|Y_k\|_p/\|Y_k\|_2 = O(k^{\alpha(p)})
\]
where
\[ \alpha(p) = \begin{cases} \frac{(n - 2)(p - 2)}{4p} & \text{if } 2 < p < \frac{2n}{n - 2}, \\ \frac{n - 2}{2} - \frac{n - 1}{p} & \text{if } \frac{2n}{n - 2} < p \leq \infty. \end{cases} \]
From the proof of Theorem 1' and more precisely from inequality (3) it follows immediately that
\[ \|Y_k\|_p/\|Y_k\|_q = O(k^{\alpha(p)}) \]
But, if we put \( \|Y_k\|_2 \) instead of \( \|Y_k\|_q \) the bound we get for the quotient is not sharp and it does not apply to obtaining nontrivial results for Bochner-Riesz operators.
Using (4) we can get in a trivial way the bound \( O(k^{2\alpha(p)}) \) for the quotient \( \|Y_k\|_p/\|Y_k\|_q \). It is easily verified that \( 2\alpha(p) \) is less than our exponent when \( p \) is close to 2 and bigger when \( p \) is close to \( \infty \) (for \( n > 3 \)).

**REFERENCES**


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