ON THE ORLICZ-PETTIS PROPERTY IN NONLOCALLY CONVEX F-SPACES

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ABSTRACT. Recently, J. H. Shapiro showed that, contrary to the case of separable F-spaces with separating duals, the Orlicz-Pettis theorem fails for \( h_p, \ 0 < p < 1 \), and some other nonseparable F-spaces of harmonic functions. In this paper we give new, much simpler examples of F-spaces for which the Orlicz-Pettis theorem fails; namely weak-L^p sequence spaces \( \ell(p, \infty) \) for \( 0 < p < 1 \). We observe that if \( 0 < p < 1 \) then the space \( \ell(p, \infty) \) is nonseparable but separable with respect to its weak topology. Moreover, we show that the Orlicz-Pettis theorem holds for every Orlicz sequence space (even nonseparable).

1. Introduction. Let \( X = (X, \tau) \) be a topological vector space whose topological dual space separates points. We say that \( X \) has the Orlicz-Pettis Property (OPP) if every weakly subseries convergent series in \( X \) (i.e. such a series \( \sum x_n \) in \( X \) that \( \text{weak-lim}_{n \to \infty} \sum_{j=1}^n x_{k_j} \) exists for each increasing sequence \( \{k_j\} \) of positive integers) is convergent in \( (X, \tau) \). The classical Orlicz-Pettis theorem states that every Banach space has the Orlicz-Pettis Property. The reader is referred to [4] for information about the Orlicz-Pettis theorem and its importance in the development of the theory of F-spaces.

We recall that OPP has all locally convex spaces or separable F-spaces (i.e. complete metrizable t.v.s.) with separating duals. Recently, J. H. Shapiro [5] has shown that the Orlicz-Pettis theorem cannot be extended to the nonseparable case. The aim of this paper is to give new, simpler natural examples of F-spaces without OPP as well as to prove other results mentioned in the abstract.

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2. The Orlicz-Pettis Property for solid spaces. In the sequel we prefer to work with Mackey topologies instead of weak topologies. We recall that the Mackey topology of a topological vector space \( X = (X, \tau) \) is the strongest locally convex topology \( \mu = \mu(X) \) on \( X \) producing the same topological dual space as \( \tau \). If \( (X, \tau) \) is an F-space whose dual separates points, then \( \mu(X) \) coincides with the strongest locally convex topology on \( X \) which is weaker than \( \tau \). Moreover, if \( \mathcal{B} \) is a base of neighborhoods of zero for \( \tau \), then the family \( \{\text{conv}^\prime U : U \in B\} \) is a base of neighborhoods of zero for \( \mu \) (see [5, Theorem 2.9]). The space \( (X, \mu(X)) \) being locally convex has OPP. Consequently, an F-space \( (X, \tau) \) has the Orlicz-Pettis Property if and only if every \( \mu(X) \)-subseries convergent series in \( X \) is \( \tau \)-convergent.
Throughout this paper we denote by $\omega$ the space of all real sequences and by $\omega_0$ the subspace of $\omega$ consisting of all sequences with finite supports. By $e_n$ is denoted the $n$th unit vector in $\omega$ and other sequence spaces, $n = 1, 2, \ldots$. For any $x = (t_n) \in \omega$ we define $R_n x = (0, 0, \ldots, 0, t_{n+1}, t_{n+2}, \ldots)$, $n = 1, 2, \ldots$.

A subset $E$ of $\omega$ is called solid if $x \in E$ and $y \in \omega$ and $|y| \leq |x|$ implies $y \in E$. Let $X = (X, \tau)$ be a t.v.s. contained set theoretically in $\omega$ and containing $\omega_0$. We say that $X$ is solid if there is a base of neighborhoods of zero for $\tau$ consisting of solid sets.

For any solid space $(X, \tau)$ we denote by $X_a$ or $X^a_T$ the closed linear subspace of $X$ spanned by the unit vectors. Let $\mathcal{B}$ be a base of solid $\tau$-neighborhoods of zero. We observe that for any $U \in \mathcal{B}$ the set $\omega_0 + U$ is solid. Therefore, $X_a = \overline{\omega_0} = \bigcap\{\omega_0 + U : U \in \mathcal{B}\}$ is a solid space. It is obvious that the family of projections $\{R_n : n \in \mathbb{N}\}$ is equicontinuous on $X$. This immediately implies that the sequence of unit vectors $\{e_n\}$ is a basis of $X_a$. $X_a$ is solid, so the series $\sum t_n e_n$ is $\tau$-subseries convergent for any $x = (t_n) \in X_a$.

Obviously, every solid $F$-space has separating dual space. It is easily verified that the convex hull and the closure of any solid set are solid. Thus, $(X, \mu(X))$ is solid for any solid $F$-space $X$.

The above observations show that if $x = (t_n) \in X_a \setminus X^a_T$, then the series $\sum t_n e_n$ is $\mu(X)$-subseries convergent but it is not $\tau$-convergent. This proves the following

**PROPOSITION 2.1.** Let $X = (X, \tau)$ be a solid $F$-space. If $X^a_T \neq X^a_a$, then $X$ does not have the Orlicz-Pettis Property.

For the proof of our next theorem we need the following version of the more general Kalton result [3].

**LEMMA 2.2.** Let $(Y, \rho)$ be a separable $F$-space and let $\nu$ be a weaker Hausdorff vector topology on $Y$. Then any $\nu$-subseries convergent series in $Y$ is $\rho$-convergent.

**THEOREM 2.3.** If $(X, \tau)$ is an $F$-space with separating dual space, $Y$ is weakly closed separable subspace of $X$ and $X/Y$ has the Orlicz-Pettis Property, then so does $X$.

**PROOF.** Suppose that $Y$ is a separable, weakly closed (so also $\mu(X)$-closed) subspace of $X$ such that $X/Y$ has OPP. Let $\| \cdot \|$ be an $F$-norm inducing the topology $\tau$ and let $\sum x_n$ be any $\mu(X)$-subseries convergent series in $X$ which is not $\tau$-convergent. Then $\{\sum_{j=1}^n x_j\}_{n \in \mathbb{N}}$ is not a Cauchy sequence in $X$, so there is an $\varepsilon > 0$ and a pair $\{j_n\}$, $\{n\}$ of sequences of positive integers such that $j_1 < l_1 < j_2 < l_2 < \cdots$ and $\|\sum_{j=j_n}^{l_n} x_j\| > \varepsilon$ for $n = 1, 2, \ldots$. Let $y_n = \sum_{j=j_n}^{l_n} x_j$, $n = 1, 2, \ldots$. Then the series $\sum y_n$ is $\mu(X)$-subseries convergent. The canonical quotient mapping $Q : X \to X/Y$ is $(\mu(X), \mu(X/Y))$-continuous, so the series $\sum Q(y_n)$ is $\mu(X/Y)$-subseries convergent. $X/Y$ has OPP, thus the series $\sum Q(y_n)$ is $\tau/Y$-subseries convergent. Passing to a subsequence we may assume that $\sum \|Q(y_n)\| < \infty$, where $\| \cdot \|_1$ is the quotient $F$-norm of $\| \cdot \|$. Therefore, there is a pair of sequences $\{u_n\} \subset Y$ and $\{v_n\} \subset X$ such that $y_n = u_n + v_n$ and $\sum \|v_n\| < \infty$. The series $\sum v_n$ being absolutely convergent is both $\tau$- and $\mu$-subseries convergent in $X$. Consequently, the series $\sum u_n$ is $\mu$-subseries convergent in $X$. However, the space $Y$ is $\mu$-closed in $X$, so the series $\sum u_n$ is $\mu$-subseries convergent in $Y$. $(Y, \tau|_Y)$ is a separable $F$-space and $\mu|_Y$ is a Hausdorff vector topology on $Y$ which is weaker than $\tau|_Y$. By Lemma...
2.2 the series \( \sum u_n \) is \( \tau \)-convergent. Finally, the series \( \sum y_n \) is \( \tau \)-convergent. This contradicts the fact that \( \|y_n\| > \varepsilon \) for \( n = 1, 2, \ldots \).

**COROLLARY 2.4.** Every Orlicz sequence space has the Orlicz-Pettis Property.

**PROOF.** Let \( l_\varphi \) be an Orlicz sequence space and let \( Y = (l_\varphi)_a \). Then the quotient space \( l_\varphi / Y \) equipped with the canonical quotient \( F \)-norm is a Banach space (see [2, Proposition 2.1]). Therefore, \( l_\varphi / Y \) has OPP and, obviously, \( Y \) is weakly closed in \( l_\varphi \). Moreover, \( Y \) is separable, so the result directly follows from Theorem 2.3.

2. Weak-\( L_p \) sequence spaces. For any sequence \( x = (t_n) \in \omega \) tending to zero we denote by \( x^* = (t_n^*) \) the nonincreasing rearrangement of the sequence \( |x| = (|t_n|) \).

If \( 0 < p < \infty \) then \( l(p, \infty) \) is the space of all sequences \( x = (t_n) \in c_0 \) such that \( \|x\|_{p, \infty} = \sup \{n^{1/p} t_n^*: \ n \in \mathbb{N} \} < \infty \).

It is easy to prove that the family of sets \( U_\varepsilon = \{x \in l(p, \infty) : \|x\|_{p, \infty} \leq \varepsilon \} \), \( \varepsilon > 0 \), is a base of neighborhoods of zero, consisting of solid sets for the unique complete, metrizable vector topology \( \lambda_{p, \infty} \) on \( l(p, \infty) \). Thus, the space \( (l(p, \infty), \lambda_{p, \infty}) \) is a solid \( F \)-space (see [1] for more details).

**THEOREM 3.1.** If \( 0 < p \leq 1 \) then \( l(p, \infty) \) does not have the Orlicz-Pettis Property.

**PROOF.** If \( 0 < p < 1 \) then the result immediately follows from Proposition 2.1 and [1, Theorem 4]. Indeed, M. Cwikel essentially showed that every continuous linear functional on \( l(p, \infty) \), \( 0 < p < 1 \), vanishing on \( \omega_0 \) is identically equal to zero, so \( \omega_0^* = l(p, \infty) \). However, \( (n^{-1/p}) \notin l(p, \infty)_a \) because the series \( \sum n^{-1/p} e_n \) is not \( \lambda_{p, \infty^}\)-convergent in \( l(p, \infty) \).

If \( p = 1 \) then the situation is somewhat more involved. Now, the series \( \sum n^{-1} e_n \) is not \( \mu \)-subseries convergent (see Remarks 3.2). However, it is still possible to find a sequence \( x = (t_n) \) in \( l(p, \infty) \) such that

(i) the series \( \sum t_n e_n \) is not \( \lambda_{1, \infty^}\)-convergent,

(ii) \( x \in l(1, \infty)_a \).

(iii) \( x \in \omega_0 + \text{conv } U_\varepsilon \) for any \( \varepsilon > 0 \).

We construct inductively an increasing sequence \( \{n_k\}_{k=0}^\infty \) of nonnegative integers such that

\[
\frac{1}{j} \sum_{i=1}^{j} \left( n_{k-1} + i \frac{n_k - n_{k-1}}{j} \right)^{-1} > \left( \frac{1}{2} \sum_{i=1}^{j} \frac{1}{i} \right) n_k^{-1}
\]

for \( j = 1, 2, \ldots, k, \ k = 1, 2, \ldots, n_0 = 0, \)

\[k! \text{ divides } n_k - n_{k-1} \quad \text{for } k = 1, 2, \ldots, \]

The above construction is possible because

\[
\lim_{t \to \infty} \frac{1}{j} \sum_{i=1}^{j} t \left( a + i \frac{t - a}{j} \right)^{-1} = \sum_{i=1}^{j} \frac{1}{i}
\]

for any \( a > 0, \ j \in \mathbb{N} \).
Let us denote \( I(k) = \{n_k - 1 + 1, \ldots, n_k\} \) for \( k = 1, 2, \ldots \). We define \( x = (t_n) \in c_0 \) taking \( t_n = n_k^{-1} \) for \( n \in I(k) \), \( k, n \in \mathbb{N} \). The sequence \( x \) is nonincreasing, positive, 
\[ nt_n \leq 1 \] and \( n_k t_k = 1 \) for \( n, k = 1, 2, \ldots \). This implies that \( \|R_n x\|_{1, \infty} = 1 \) for \( n = 1, 2, \ldots \), so \( x \in l(1, \infty) \) and the series \( \sum t_n x_n \) is not \( \lambda_{1, \infty} \)-convergent. The proof will be finished if we show that \( x \) satisfies (iii).

Fix \( \varepsilon > 0 \). Choose \( j \in \mathbb{N} \) such that
\[
c_j := \frac{1}{2} \sum_{i=1}^{j} \frac{1}{i} > \varepsilon^{-1}.
\]
Then, by (b), \( j \) divides \( n_k - n_{k-1} \) for \( k \geq j \). Let
\[
l_{k,i} = n_{k-1} + i \frac{n_k - n_{k-1}}{j}
\]
for \( i = 0, 1, \ldots, j \), \( k = j, j + 1, \ldots \), and
\[
I(k,i) = \{l_{k,i} + 1, \ldots, l_{k,i+1}\}
\]
for \( i = 0, 1, \ldots, j - 1 \), \( k = j, j + 1, \ldots \). We define \( y_m = (s_{m,n})_{n=1}^{\infty} \in \omega \), \( m = 0, 1, \ldots, j - 1 \), taking
\[
s_{m,n} = \begin{cases} 
(c_j l_{k,i+1})^{-1} & \text{if } n \in I(k, (m + i) \mod j) \text{ for some } \\
0 & \text{otherwise.}
\end{cases}
\]
It is easily seen that \( \sup \{n s_{m,n}^* : n \in \mathbb{N}\} \leq \varepsilon \), so \( y_m \in U_\varepsilon \) for \( m = 0, 1, \ldots, j - 1 \). If \( n \in I(k,i) \) for some \( i = 0, 1, \ldots, j - 1 \), \( k = j, j + 1, \ldots \), then by (a)
\[
\frac{1}{j} \sum_{m=0}^{j-1} s_{m,n} = \frac{1}{j} \sum_{i=1}^{j} \left[ c_j \left( n_{k-1} + i \frac{n_k - n_{k-1}}{j} \right) \right]^{-1} \geq n_k^{-1} = t_n.
\]
We define \( z = (s_n) \in \omega_0 \) taking \( s_n = t_n \) for \( n = 1, 2, \ldots, n_j - 1 \), and \( s_n = 0 \) for \( n > n_j - 1 \). Therefore, by (c)
\[
|x| < z + \frac{1}{j} \sum_{m=0}^{j-1} y_m \in \omega_0 + \text{conv } U_\varepsilon.
\]
This implies (iii) because the set \( \omega_0 + \text{conv } U_\varepsilon \) is solid.

REMARKS 3.2. (a) We have just observed that the sequence \( (n^{-1/p}) \) does not belong to \( l(p, \infty)_a \), \( 0 < p < \infty \). Essentially, it is easy to prove that
\[
l(p, \infty)_a = \{x = (t_n) \in c_0 : \lim n^{1/p} t_n^* = 0\}.
\]
(b) We have noticed that if \( 0 < p < 1 \) then \( \omega_0 \) is weakly dense in \( l(p, \infty) \). Therefore, \( l(p, \infty) \) for \( 0 < p < 1 \) are new examples of \( F \)-spaces which are nonseparable but their weak topologies are Hausdorff and separable (see also [5]).
(c) \( l(1, \infty) \) is nonseparable in its Mackey (so also weak) topology. Indeed, it is easy to see that the functional
\[
q(x) = \sup_n \frac{\sum_{i=1}^{n} t_i^*}{\sum_{i=1}^{n} (1/i)^*}, \quad x = (t_i) \in c_0,
\]
is a continuous norm on $l(1, \infty)$. There is an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $1/2 \leq q(z_k) \leq 1$, where $z_k = \sum_{n=n_k}^{n_{k+1}} n^{-1} e_n$, $n_0 = 0$, $k = 1, 2, \ldots$. Now we observe that the mapping $l_\infty \ni (s_n) \mapsto \sum s_n z_n$ (the convergence in the product topology) is an isomorphism of $l_\infty$ into $l(1, \infty)$ equipped with the topology $\rho$ induced by $q$. $\rho$ is weaker than the Mackey topology $\mu$ of $l(1, \infty)$, so the space $(l(1, \infty), \mu)$ is nonseparable.

Let us note that the series $\sum n^{-1} e_n$ is not $\rho$-(so also $\mu$-) convergent.

(d) The author does not know whether the topology $\rho$ defined above coincides with the Mackey topology of $l(1, \infty)$. We have observed only that $\rho \leq \mu(l(1, \infty))$. However, let us notice that $\rho$ induces on $l(1, \infty)_a$ the Mackey topology of $l(1, \infty)_a$.

Indeed, it is easy to prove that if a sequence $x = (t_n)$ is an extreme point of the compact, convex set $B_n = B \cup \text{span}\{e_1, e_2, \ldots, e_n\}$ where $B = \{x \in l(1, \infty) : q(x) \leq 1\}$, $n \in \mathbb{N}$, then $x^* = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$. Therefore, every extreme point of $B_n$ belongs to the unit ball $U$ of $l(1, \infty)_a$. Consequently, $\text{conv} U \supset B \cap \omega_0$. This, the density of $\omega_0$ in $l(1, \infty)_a$, and the homogeneity of the functionals $\|\cdot\|_{1, \infty}$ and $q$ imply that the topology induced by $\rho$ on $l(1, \infty)_a$ is a stronger that $\mu(l(1, \infty)_a)$.

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