PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 3, November 1987

ON THE ORLICZ-PETTIS PROPERTY IN NONLOCALLY CONVEX F-SPACES

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(Communicated by William J. Davis)

ABSTRACT. Recently, J. H. Shapiro showed that, contrary to the case of separable F-spaces with separating duals, the Orlicz-Pettis theorem fails for h_p, $0 < p < 1$, and some other nonseparable F-spaces of harmonic functions. In this paper we give new, much simpler examples of F-spaces for which the Orlicz-Pettis theorem fails; namely weak-L_p sequence spaces $l(p, \infty)$ for $0 < p \leq 1$. We observe that if $0 < p < 1$ then the space $l(p, \infty)$ is nonseparable but separable with respect to its weak topology. Moreover, we show that the Orlicz-Pettis theorem holds for every Orlicz sequence space (even nonseparable).

1. Introduction. Let $X = (X, \tau)$ be a topological vector space whose topological dual space separates points. We say that X has the *Orlicz-Pettis Property* (OPP) if every weakly subseries convergent series in X (i.e. such a series $\sum x_n$ in X that weak-$\lim_{n\to\infty} \sum_{j=1}^{n} x_{k_j}$ exists for each increasing sequence $\{k_j\}$ of positive integers) is convergent in (X, τ). The classical Orlicz-Pettis theorem states that every Banach space has the Orlicz-Pettis Property. The reader is referred to [**4**] for information about the Orlicz-Pettis theorem and its importance in the development of the theory of F-spaces.

We recall that OPP has all locally convex spaces or separable F-spaces (i.e. complete metrizable t.v.s.) with separating duals. Recently, J. H. Shapiro [**5**] has shown that the Orlicz-Pettis theorem cannot be extended to the nonseparable case. The aim of this paper is to give new, simpler natural examples of F-spaces without OPP as well as to prove other results mentioned in the abstract.

I wish to thank Lech Drewnowski and Augustyn Ortynski for helpful remarks on this material.

2. The Orlicz-Pettis Property for solid spaces. In the sequel we prefer to work with Mackey topologies instead of weak topologies. We recall that the Mackey topology of a topological vector space $X = (X, \tau)$ is the strongest locally convex topology $\mu = \mu(X)$ on X producing the same topological dual space as τ. If (X, τ) is an F-space whose dual separates points, then $\mu(X)$ coincides with the strongest locally convex topology on X which is weaker than τ. Moreover, if $\mathcal{B}$ is a base of neighborhoods of zero for τ, then the family $\{\overline{\mathrm{conv}}^{\tau} U : U \in \mathcal{B}\}$ is a base of neighborhoods of zero for μ (see [**5**, Theorem 2.9]). The space $(X, \mu(X))$ being locally convex has OPP. Consequently, an F-space (X, τ) has the Orlicz-Pettis Property if and only if every $\mu(X)$-subseries convergent series in X is τ-convergent.

Received by the editors April 24, 1986 and, in revised form, July 20, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46A06, 46A35, 45A45.
Key words and phrases. The Orlicz-Pettis theorem, weak-L_p spaces, Orlicz spaces.

Throughout this paper we denote by ω the space of all real sequences and by ω_0 the subspace of ω consisting of all sequences with finte supports. By e_n is denoted the nth unit vector in ω and other sequence spaces, $n = 1, 2, \ldots$. For any $x = (t_n) \in \omega$ we define $R_n x = (0, 0, \ldots, 0, t_{n+1}, t_{n+2}, \ldots)$, $n = 1, 2, \ldots$.

A subset E of ω is called *solid* if $x \in E$ and $y \in \omega$ and $|y| \leq |x|$ implies $y \in E$. Let $X = (X, \tau)$ be a t.v.s. contained set theoretically in ω and containing ω_0. We say that X is *solid* if there is a base of neighborhoods of zero for τ consisting of solid sets.

For any solid space (X, τ) we denote by X_a or X_a^τ the closed linear subspace of X spanned by the unit vectors. Let $\mathcal{B}$ be a base of solid τ-neighborhoods of zero. We observe that for any $U \in \mathcal{B}$ the set $\omega_0 + U$ is solid. Therefore, $X_a = \overline{\omega}_0^\tau = \bigcap\{\omega_0 + U \colon U \in \mathcal{B}\}$ is a solid space. It is obvious that the family of projections $\{R_n \colon n \in \mathbf{N}\}$ is equicontinuous on X. This immediately implies that the sequence of unit vectors $\{e_n\}$ is a basis of X_a. X_a is solid, so the series $\sum t_n e_n$ is τ-subseries convergent for any $x = (t_n) \in X_a$.

Obviously, every solid F-space has separating dual space. It is easliy verified that the convex hull and the closure of any solid set are solid. Thus, $(X, \mu(X))$ is solid for any solid F-space X.

The above observations show that if $x = (t_n) \in X_a^\mu \backslash X_a^\tau$, then the series $\sum t_n e_n$ is $\mu(X)$-subseries convergent but it is not τ-convergent. This proves the following

PROPOSITION 2.1. *Let $X = (X, \tau)$ be a solid F-space. If $X_a^\tau \neq X_a^\mu$, then X does not have the Orlicz-Pettis Property.*

For the proof of our next theorem we need the following version of the more general Kalton result [**3**].

LEMMA 2.2. *Let (Y, ρ) be a separable F-space and let ν be a weaker Hausdorff vector topology on Y. Then any ν-subseries convergent series in Y is ρ-convergent.*

THEOREM 2.3. *If (X, τ) is an F-space with separating dual space, Y is weakly closed separable subspace of X and X/Y has the Orlicz-Pettis Property, then so does X.*

PROOF. Suppose that Y is a separable, weakly closed (so also $\mu(X)$-closed) subspace of X such that X/Y has OPP. Let $\|\cdot\|$ be an F-norm inducing the topology τ and let $\sum x_n$ be any $\mu(X)$-subseries convergent series in X which is not τ-convergent. Then $\{\sum_{j=1}^n x_j\}_{n\in\mathbf{N}}$ is not a Cauchy sequence in X, so there is an $\varepsilon > 0$ and a pair $\{j_n\}$, $\{l_n\}$ of sequences of positive integers such that $j_1 < l_1 < j_2 < l_2 < \cdots$ and $\|\sum_{j=j_n}^{l_n} x_j\| > \varepsilon$ for $n = 1, 2, \ldots$. Let $y_n = \sum_{j=j_n}^{l_n} x_j$, $n = 1, 2, \ldots$. Then the series $\sum y_n$ is $\mu(X)$-subseries convergent. The canonical quotient mapping $Q \colon X \to X/Y$ is $(\mu(X), \mu(X/Y))$-continuous, so the series $\sum Q(y_n)$ is $\mu(X/Y)$-subseries convergent. X/Y has OPP, thus the series $\sum Q(y_n)$ is τ/Y-subseries convergent. Passing to a subsequence we may assume that $\sum \|Q(y_n)\|_1 < \infty$, where $\|\cdot\|_1$ is the quotient F-norm of $\|\cdot\|$. Therefore, there is a pair of sequences $\{u_n\} \subset Y$ and $\{v_n\} \subset X$ such that $y_n = u_n + v_n$ and $\sum \|v_n\| < \infty$. The series $\sum v_n$ being absolutely convergent is both τ- and μ-subseries convergent in X. Consequently, the series $\sum u_n$ is μ-subseries convergent in X. However, the space Y is μ-closed in X, so the series $\sum u_n$ is μ-subseries convergent in Y. $(Y, \tau|_Y)$ is a separable F-space and $\mu|_Y$ is a Hausdorff vector topology on Y which is weaker than $\tau|_Y$. By Lemma

2.2 the series $\sum u_n$ is τ-convergent. Finally, the series $\sum y_n$ is τ-convergent. This contradicts the fact that $\|y_n\| > \varepsilon$ for $n = 1, 2, \ldots$.

COROLLARY 2.4. *Every Orlicz seqence space has the Orlicz-Pettis Property.*

PROOF. Let $l_\varphi$be an Orlicz sequence space and let $Y = (l_\varphi)_a$. Then the quotient space l_φ/Y equipped with the canonical quotient F-norm is a Banach space (see [**2**, Proposition 2.1]). Therefore, l_φ/Y has OPP and, obviously, Y is weakly closed in l_φ. Moreover, Y is separable, so the result directly follows from Theorem 2.3.

2. Weak-L_p sequence spaces. For any sequence $x = (t_n) \in \omega$ tending to zero we denote by $x^* = (t_n^*)$ the nonincreasing rearrangement of the sequence $|x| = (|t_n|)$.

If $0 < p < \infty$ then $l(p, \infty)$ is the space of all sequences $x = (t_n) \in c_0$ such that $\|x\|_{p,\infty} = \sup\{n^{1/p} t_n^* \colon n \in \mathbf{N}\} < \infty$.

It is easy to prove that the family of sets $U_\varepsilon = \{x \in l(p, \infty) \colon \|x\|_{p,\infty} \leq \varepsilon\}$, $\varepsilon > 0$, is a base of neighborhoods of zero, consisting of solid sets for the unique complete, metrizable vector topology $\lambda_{p,\infty}$ on $l(p, \infty)$. Thus, the space $(l(p, \infty), \lambda_{p,\infty})$ is a solid F-space (see [**1**] for more details).

THEOREM 3.1. *If $0 < p \leq 1$ then $l(p, \infty)$ does not have the Orlicz-Pettis Property.*

PROOF. If $0 < p < 1$ then the result immediately follows from Proposition 2.1 and [**1**, Theorem 4]. Indeed, M. Cwikel essentially showed that every continuous linear functional on $l(p, \infty)$, $0 < p < 1$, vanishing on ω_0 is identically equal to zero, so $\bar{\omega}_0^\mu = l(p, \infty)$. However, $(n^{-1/p}) \notin l(p, \infty)_a$ because the series $\sum n^{-1/p} e_n$ is not $\lambda_{p,\infty}$-convergent in $l(p, \infty)$.

If $p = 1$ then the situation is somewhat more involved. Now, the series $\sum n^{-1} e_n$ is not μ-subseries convergent (see Remarks 3.2). However, it is still possible to find a sequence $x = (t_n)$ in $l(p, \infty)$ such that

(i) the series $\sum t_n e_n$ is not $\lambda_{1,\infty}$-convergent,

(ii) $x \in l(1, \infty)_a^\mu$.

(ii) is equivalent to

(iii) $x \in \omega_0 + \operatorname{conv} U_\varepsilon$ for any $\varepsilon > 0$.

We construct inductively an increasing sequence $\{n_k\}_{k=0}^\infty$ of nonnegative integers such that

$$\text{(a)} \qquad \frac{1}{j} \sum_{i=1}^{j} \left(n_{k-1} + i \frac{n_k - n_{k-1}}{j}\right)^{-1} > \left(\frac{1}{2} \sum_{i=1}^{j} \frac{1}{i}\right) n_k^{-1}$$

for $j = 1, 2, \ldots, k$, $k = 1, 2, \ldots, n_0 = 0$,

$$\text{(b)} \qquad k! \text{ divides } n_k - n_{k-1} \quad \text{for } k = 1, 2, \ldots.$$

The above construction is possible because

$$\lim_{t \to \infty} \frac{1}{j} \sum_{i=1}^{j} t \left(a + i \frac{t - a}{j}\right)^{-1} = \sum_{i=1}^{j} \frac{1}{i}$$

for any $a > 0$, $j \in \mathbf{N}$.

Let us denote $I(k) = \{n_{k-1}+1, \ldots, n_k\}$ for $k = 1, 2, \ldots$. We define $x = (t_n) \in c_0$ taking $t_n = n_k^{-1}$ for $n \in I(k)$, $k, n \in \mathbf{N}$. The sequence x is nonincreasing, positive, $nt_n \leq 1$ and $n_k t_k = 1$ for $n, k = 1, 2, \ldots$. This implies that $\|R_n x\|_{1,\infty} = 1$ for $n = 1, 2, \ldots$, so $x \in l(1, \infty)$ and the series $\sum t_n x_n$ is not $\lambda_{1,\infty}$-convergent. The proof will be finished if we show that x satisifies (iii).

Fix $\varepsilon > 0$. Choose $j \in \mathbf{N}$ such that

$$c_j := \frac{1}{2} \sum_{i=1}^{j} \frac{1}{i} > \varepsilon^{-1}.$$

Then, by (b), j divides $n_k - n_{k-1}$ for $k \geq j$. Let

$$l_{k,i} = n_{k-1} + i \frac{n_k - n_{k-1}}{j}$$

for $i = 0, 1, \ldots, j$, $k = j, j+1, \ldots$, and

$$I(k, i) = \{l_{k,i} + 1, \ldots, l_{k,i+1}\}$$

for $i = 0, 1, \ldots, j-1$, $k = j, j+1, \ldots$. We define $y_m = (s_{m,n})_{n=1}^{\infty} \in \omega$, $m = 0, 1, \ldots, j-1$, taking

$$s_{m,n} = \begin{cases} (c_j l_{k,i+1})^{-1} & \text{if } n \in I(k, (m+i)_{\bmod (j)}) \text{ for some} \\ & i = 0, 1, \ldots, j-1, \ k = j, j+1, \ldots, \\ 0 & \text{otherwise} . \end{cases}$$

It is easily seen that $\sup\{ns_{m,n}^* : n \in \mathbf{N}\} \leq \varepsilon$, so $y_m \in U_\varepsilon$ for $m = 0, 1, \ldots, j-1$. If $n \in I(k, i)$ for some $i = 0, 1, \ldots, j-1$, $k = j, j+1, \ldots$, then by (a)

$$\text{(c)} \qquad \frac{1}{j} \sum_{m=0}^{j-1} s_{m,n} = \frac{1}{j} \sum_{i=1}^{j} \left[c_j \left(n_{k-1} + i \frac{n_k - n_{k-1}}{j} \right) \right]^{-1} \geq n_k^{-1} = t_n.$$

We define $z = (s_n) \in \omega_0$ taking $s_n = t_n$ for $n = 1, 2, \ldots, n_{j-1}$, and $s_n = 0$ for $n > n_{j-1}$. Therefore, by (c)

$$|x| < z + \frac{1}{j} \sum_{m=0}^{j-1} y_m \in \omega_0 + \operatorname{conv} U_\varepsilon.$$

This implies (iii) because the set $\omega_0 + \operatorname{conv} U_\varepsilon$ is solid.

REMARKS 3.2. (a) We have just observed that the sequence $(n^{-1/p})$ does not belong to $l(p, \infty)_a$, $0 < p < \infty$. Essentially, it is easy to prove that

$$l(p, \infty)_a = \{x = (t_n) \in c_0 \colon \lim n^{1/p} t_n^* = 0\}.$$

(b) We have noticed that if $0 < p < 1$ then ω_0 is weakly dense in $l(p, \infty)$. Therefore, $l(p, \infty)$ for $0 < p < 1$ are new examples of F-spaces which are *nonseparable* but their weak topologies are Hausdorff and *separable* (see also [**5**]).

(c) $l(1, \infty)$ *is nonseparable in its Mackey* (*so also weak*) *topology*. Indeed, it is easy to see that the functional

$$q(x) = \sup_n \frac{\sum_{i=1}^{n} t_i^*}{\sum_{i=1}^{n} (1/i)}, \qquad x = (t_i) \in c_0,$$

is a continuous norm on $l(1,\infty)$. There is an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $1/2 \leq q(z_k) \leq 1$, where $z_k = \sum_{n=n_k+1}^{n_{k+1}} n^{-1}e_n$, $n_0 = 0$, $k = 1, 2, \ldots$. Now we observe that the mapping $l_\infty \ni (s_n) \longmapsto \sum s_n z_n$ (the convergence in the product topology) is an isomorphism of l_∞ into $l(1,\infty)$ equipped with the topology ρ induced by q. ρ is weaker than the Mackey topology μ of $l(1,\infty)$, so the space $(l(1,\infty),\mu)$ is nonseparable.

Let us note that the series $\sum n^{-1}e_n$ is not ρ- (so also μ-) convergent.

(d) The author does not know whether the topology ρ defined above coincides with the Mackey topology of $l(1,\infty)$. We have observed only that $\rho \leq \mu(l(1,\infty))$. However, let us notice that *ρ induces on $l(1,\infty)_a$ the Mackey topology of $l(1,\infty)_a$*. Indeed, it is easy to prove that if a sequence $x = (t_n)$ is an extreme point of the compact, convex set $B_n = B \cup \operatorname{span}\{e_1, e_2, \ldots, e_n\}$ where $B = \{x \in l(1,\infty) \colon q(x) \leq 1\}$, $n \in \mathbf{N}$, then $x^* = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$. Therefore, every extreme point of B_n belongs to the unit ball U of $l(1,\infty)_a$. Consequently, $\overline{\operatorname{conv}}\, U \supset B \cap \omega_0$. This, the density of ω_0 in $l(1,\infty)_a$, and the homogeneity of the functionals $\|\cdot\|_{1,\infty}$ and q imply that the topolgy induced by ρ on $l(1,\infty)_a$ is a stronger that $\mu(l(1,\infty)_a)$.

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