CORRIGENDUM TO “PARTITIONS AND DIAMOND”

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(Communicated by Thomas J. Jech)

We recall that given an uncountable cardinal \( \kappa \), \( \diamond \kappa \) asserts the existence of a family \( s_\alpha \subseteq \alpha \), \( \alpha < \kappa \), such that the set \( \{ \alpha < \kappa : A \cap \alpha = s_\alpha \} \) is stationary in \( \kappa \) for all \( A \subseteq \kappa \).

It occurred to us that the implication (iv) \( \rightarrow \) (i) in Proposition 4 of [4] needs not hold for \( \kappa > \omega_1 \). In that case, a slight modification of the original argument yields that (i)-(iii) in the proposition are indeed equivalent, and that they are equivalent to this stronger form of (iv): There exists a family \( Z_\alpha \in (\kappa)^2 \), \( \alpha < \kappa \), such that the diagonal intersection \( \Delta\{Z_\alpha(h(\alpha)) : \alpha < \kappa \} \) is stationary for every \( h \in 2^\kappa \).

The incorrect proof relied on the claim, p. 39 of [1], that assuming \( \kappa \) is regular, \( \diamond \kappa \) follows from the existence of a sequence \( s_\alpha \subseteq \alpha \), \( \alpha < \kappa \), such that whenever \( A \subseteq \kappa \), there is an infinite \( \alpha \) with \( A \cap \alpha = s_\alpha \). That this is indeed the case when \( \kappa = \omega_1 \) was shown by Devlin in [2]. However, this cannot be true in general, as it would imply that \( \diamond \kappa \) holds whenever \( \kappa \) is regular and \( \diamond \lambda \) holds for some uncountable cardinal \( \lambda < \kappa \). It is nevertheless possible to generalize Devlin’s result as follows:

**PROPOSITION.** Let \( \lambda, \kappa \) be infinite cardinals with \( 2^\lambda \geq \kappa \). Assume there are \( P_\alpha \), \( \alpha < \kappa \), such that each \( P_\alpha \) is a collection of size \( \leq |\alpha| \) of subsets of \( \alpha \), and that for every \( A \subseteq \kappa \), there is an \( \alpha \geq \lambda \) with \( A \cap \alpha \in P_\alpha \). Then \( \diamond \kappa \) holds.

**PROOF.** Let \( P_\alpha \), \( \alpha < \kappa \), be as in the statement of the proposition. By a well-known result of Kunen [3], it is enough to show that \( \kappa \) is regular and there exist \( Q_\alpha \), \( \alpha < \kappa \), such that each \( Q_\alpha \) is a collection of size \( \leq |\alpha| \) of subsets of \( \alpha \), and the set \( \{ \alpha : B \cap \alpha \in Q_\alpha \} \) is stationary in \( \kappa \) for all \( B \subseteq \kappa \). Define functions \( i, j \) from \( \kappa \) to \( \kappa \) by letting \( i(\alpha) = \lambda \alpha \) and \( j(\alpha) = 2\alpha \). Set \( N = \kappa - j[\kappa] \). For each \( \alpha \in [\lambda, \kappa) \), denote by \( Q_\alpha \) the collection of all those subsets \( D \) of \( \alpha \) such that there are \( \beta < \lambda \) and \( H \in P_{\alpha + \beta} \) with \( D = j^{-1}[H \cap \alpha] \). Fix \( B \subseteq \kappa \), and let \( C \) be a closed unbounded subset of \( \kappa \). Let \( c_\beta, 0 < \beta < \rho \), be the increasing enumeration of the set \( C \cap [\kappa - \rho, \kappa - 2) \), and put \( c_0 = 0 \). Choose \( E_\beta \subseteq N \) with \( \beta < \rho \), with the following properties: \( E_\beta \subseteq N \cap [c_\beta, c_\beta + \lambda) \), and \( E_\beta \neq H \cap N \cap [c_\beta, c_\beta + \lambda) \) whenever \( H \in P_\alpha \) with \( \alpha \in [c_\beta + \lambda, c_{\beta + 1}) \). Then set \( A = j[B] \cup \left( \bigcup_{\beta < \rho} E_\beta \right) \). Select \( \alpha \geq \lambda \) with \( A \cap \alpha \in P_\alpha \).

It is easily verified that \( B \cap c_\beta \in Q_{c_\beta} \), where \( \beta \) is such that \( \alpha \in [c_\beta, c_\beta + \lambda) \). It only remains to show that \( \kappa \) is regular. First note that \( 2^\mu = \kappa \) holds for every cardinal \( \mu \in [\lambda, \kappa) \), as the set \( \bigcup_{\alpha \geq \lambda} Q_\alpha \) has size \( \kappa \). Thus \( cf \kappa = cf(2^\mu) > \mu \) for all \( \mu \in [\lambda, \kappa) \), and consequently \( cf \kappa = \kappa \).

We remark that Theorem 4 (where (b) should read \( \diamond_{\lambda^+}(\lambda^+ - \lambda) \)) of [5] is the special case of our result when \( \kappa = \lambda^+ \). Also, a straightforward modification of the proof of the proposition yields the implication c) \( \rightarrow \) a) of Theorem 3 of [5].
Finally, we would like to point out that in [4], Proposition 3 easily follows from Proposition 1, by the following remark: given a cardinal $\kappa$ and a family $A_\alpha \in [\kappa]^\kappa$, $\alpha < \kappa$, with the property that $A_\alpha \subseteq A_\beta$ whenever $\beta < \alpha$, there is a $B \in [\kappa]^\kappa$ such that $|B - A_\alpha| < \kappa$ for all $\alpha$.

REFERENCES


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