CONTINUOUSLY HOMOGENEOUS CONTINUA
AND THEIR ARC COMPONENTS

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(Communicated by Doug W. Curtis)

Abstract. Let $X$ be a continuously homogeneous Hausdorff continuum. We prove that if there is a sequence $A_1, A_2, \ldots$ of its arc components with $X = \text{cl} A_1 \cup \text{cl} A_2 \cup \cdots$, and there is an arc component of $X$ with nonempty interior, then $X$ is arcwise connected. As consequences and applications we get: (1) if $X$ is the countable union of arcwise connected continua, then $X$ is arcwise connected; (2) if $X$ is nondegenerate and metric, the number of its arc components is countable and it contains no simple triod, then it is either an arc or a simple closed curve; and, in particular, (3) an arc is the only nondegenerate continuously homogeneous arc-like metric continuum with countably many arc components.

Introduction. Recall that a space $X$ is said to be continuously homogeneous if for every two points $x, y \in X$ there is a continuous surjection $f: X \to X$ with $f(x) = y$. This notion is due to D. P. Bellamy, and also, in a more general version, to J. J. Charatonik (see [C]). The purpose of this paper is to prove that

$$\text{if a Hausdorff continuum } X \text{ is continuously homogeneous and there is a sequence } A_1, A_2, \ldots \text{ of its arc components such that } X = \text{cl} A_1 \cup \text{cl} A_2 \cup \cdots, \text{ and there is an arc component of } X \text{ with nonempty interior, then } X \text{ is arcwise connected.}$$

As can be seen, this fact is a strengthening of the result 3 of [K2, p. 270]. P. Krupski suggested there that such an improvement (in a somewhat weaker version—see Remark in [K2, p. 271]) might be true.

Actually, conclusion (1) is one of the applications of Theorem 1 below. The notion of an arc component is replaced in this theorem by the concept of a $\mathcal{K}$-component defined as follows (compare [PI]). Let $X$ be a space and $\mathcal{K}$ be an arbitrary family of subcontinua of $X$ satisfying the two following conditions:

1. if $K = K_1 \cup K_2$ with $K_1, K_2 \in \mathcal{K}$ and $K_1 \cap K_2 = \emptyset$, then $K \in \mathcal{K}$,
2. if $K \in \mathcal{K}$ and $f: X \to X$ is a continuous surjection, then $f(K) \in \mathcal{K}$.

A set $Y \subset X$ is said to be $\mathcal{K}$-connected if each two of its points lie in a subcontinuum of it belonging to $\mathcal{K}$. The maximal $\mathcal{K}$-connected subsets of the space

Received by the editors January 28, 1986 and, in revised form, August 6, 1986.
1980 Mathematics Subject Classification (1985 Revision). Primary 54F20; Secondary 54F65.
Key words and phrases. Continuous homogeneity, covering sequence, Hausdorff continuum, $\mathcal{K}$-component, simple triod.
are called its $\mathcal{K}$-components. It can easily be seen that $\mathcal{K}$-components are arc components when $\mathcal{K}$ is the family of all locally connected metric subcontinua of $X$. Now we are ready to formulate the theorem.

1. **Theorem.** If a Hausdorff continuum $X$ is continuously homogeneous and there is a sequence $A_1, A_2, \ldots$ of its $\mathcal{K}$-components such that $X = \text{cl} A_1 \cup \text{cl} A_2 \cup \cdots$, and there is a $\mathcal{K}$-component of $X$ with nonempty interior, then $X$ is $\mathcal{K}$-connected.

To give some possible applications of this theorem, other than one of (1), note that if:

- $X$ is planable and $\mathcal{K}$ is the family of all $\delta$-connected subcontinua of $X$ (see [H1, H3 and H2]), or
- $\mathcal{K}$ is the family of all metric subcontinua of $X$ with index of local disconnectivity less than $\alpha$, for $\alpha < \omega$ (for the definition see [P2, Chapter IV]), or
- $\mathcal{K}$ is the family of all weakly chainable metric subcontinua of $X$ (see [L]), or
- $\mathcal{K}$ is the family of all subcontinua of $X$ that are continuous images of Hausdorff arcs,

then essentially different kinds of $\mathcal{K}$-connectedness are obtained.

In this way, i.e., by considering an arbitrary $\mathcal{K}$-connectedness instead of the arc connectedness only, we may extend a number of results concerning continuous homogeneity (e.g. Propositions 4 and 5 of [K1, p. 354], 1 of [K2, p. 269], and Theorem 3 of [CG, p. 341]).

All spaces considered here are assumed to be Hausdorff. A mapping is a continuous mapping between topological spaces, a surjection is a surjective mapping. An arc is a homeomorphic image of the unit segment $[0,1]$, a Hausdorff arc is a linearly ordered continuum. Symbol $ab$ denotes an arc with ends $a$ and $b$. The union of three arcs $px$, $py$, and $pz$ is called a simple triod if $px \cap py = px \cap pz = py \cap pz = \{p\}$. A point $e$ is called an end point of a space $X$ if $e \in X$ and for every two arcs $C_1, C_2 \subset X$ both containing $e$ we have either $C_1 \subset C_2$ or $C_2 \subset C_1$. The letters $\omega$ and $\omega^+$ denote the first infinite and the first uncountable ordinal, respectively. In this paper, according to [KM, p. 235], 0 is considered to be a limit ordinal.

**Covering sequences of compact spaces.** The proof of Theorem 1 makes heavy use of Lemma 3 below. In order to obtain this lemma the notion of a covering sequence of a compact space is employed. This notion is analogous to the concept of a covering sequence of a metric compactum defined in [P1]. Let $X$ be a compact space with card $X \geq \aleph_0$, and let $\Gamma$ be a limit ordinal of cardinality greater than card $X$. A sequence $\tau = \{X_\alpha\}_{\alpha < \Gamma}$ of compact subsets $X_\alpha$ of $X$ is called a covering sequence of $X$ provided for every $\alpha < \Gamma$ there is a countable ordinal $\beta$ such that $\bigcup \{X_\gamma: \alpha \leq \gamma \leq \alpha + \beta\} = X$. For every covering sequence $\tau = \{X_\alpha\}_{\alpha < \Gamma}$ we inductively define another sequence $\{D_\alpha(\tau)\}_{\alpha < \Gamma}$ of compact subsets of $X$:

- $D_0(\tau) = X$,
- $D_{\alpha+1}(\tau) = \text{cl}(D_\alpha(\tau) \setminus X_\alpha)$,
- $D_\varphi(\tau) = \bigcap \{D_\alpha(\tau): \alpha < \varphi\}$, for each limit ordinal $\varphi > 0$. 

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Note that the sequence \( \{D_a(\tau)\}_{a < \Gamma} \) is decreasing. Observe the following two properties of this sequence.

(2) For every \( a < \Gamma \) with \( D_a(\tau) \neq \emptyset \) there is \( \beta < \Omega \) such that \( D_{a+\beta}(\tau) \subseteq D_a(\tau) \).

Indeed, let \( \beta \) be a number guaranteed by the definition for the number \( a \). Then the family \( \{D_a(\tau) \cap X_\gamma : \alpha \leq \gamma \leq a + \beta\} \) covers the set \( D_a(\tau) \), thus, by the Baire theorem, one of its elements \( D_a(\tau) \cap X_{y_0} \) has nonempty interior in \( D_a(\tau) \). Therefore

\[
D_{a+\beta+1}(\tau) \subset D_{y_0+1}(\tau) = \text{cl}(D_{y_0}(\tau) \setminus X_{y_0}) \subset \text{cl}(D_a(\tau) \setminus X_{y_0}) \subseteq D_a(\tau).
\]

(3) There is an ordinal \( \beta < \Gamma \) with \( D_{\beta}(\tau) = \emptyset \).

In fact, put \( P = \{ \alpha < \Gamma : D_{a+1}(\tau) \subset D_a(\tau) \} \). Then we have \( \text{card} \, P \leq \text{card} \, X \). By (2) we see that there are only countably many ordinals between any \( \alpha \in P \) and its successor in \( P \). Therefore cardinality of the ordinal \( \sup P \) cannot exceed the cardinal \( \aleph_0 \cdot \text{card} \, P \leq \aleph_0 \cdot \text{card} \, X = \text{card} \, X \). By the assumption about the cardinality of \( \Gamma \) we have \( \text{sup} \, P < \Gamma \) which means that the sets \( D_a(\tau) \) for \( a > \sup \, P \) are identical. Thus \( D_a(\tau) = \emptyset \) for such \( a \) by (2).

The minimum ordinal \( \beta \) such that \( D_{\beta}(\tau) = \emptyset \) is denoted by \( \lambda(\tau) \).

For further purposes we define a binary operation \( * \) between covering sequences of the same space. Let \( \tau = \{X_a\}_{a < \Gamma}, \quad \psi = \{Y_a\}_{a < \Gamma} \) be covering sequences of \( X \). Then we put \( \tau \ast \psi = \{Z_a\}_{a < \Gamma}, \) where

\[
Z_{2a} = X_a, \quad Z_{2a+1} = Y_a.
\]

Obviously \( \tau \ast \psi \) is also a covering sequence of \( X \). Now we prove that

(4) \( D_\psi(\tau \ast \psi) \subset D_\psi(\tau) \cap D_\psi(\psi) \) for every limit ordinal \( \varphi < \Gamma \).

**Proof.** For \( \varphi = 0 \) it is a consequence of the definition. Assume (4) is true for a limit ordinal \( \varphi \). Thus \( D_\psi(\tau \ast \psi) \subset D_\psi(\tau) \), and, using the induction, we prove that for every natural \( n \)

\[
D_{\varphi+n}(\tau \ast \psi) \subset D_{\varphi+n}(\tau).
\]

Indeed, noting \( \varphi + 2n = 2\varphi + 2n = 2(\varphi + n) \), we see that

\[
D_{\varphi+2(n+1)}(\tau \ast \psi) \subset D_{\varphi+2n+1}(\tau \ast \psi) = \text{cl}(D_{\varphi+2n}(\tau \ast \psi) \setminus Z_{2\varphi+n})
= \text{cl}(D_{\varphi+2n}(\tau \ast \psi) \setminus Z_{2\varphi+n}) = \text{cl}(D_{\varphi+2n}(\tau \ast \psi) \setminus X_{\varphi+n})
\subset \text{cl}(D_{\varphi+n}(\tau) \setminus X_{\varphi+n}) = D_{\varphi+(n+1)}(\tau).
\]

Hence, by the definition,

\[
D_{\varphi+\omega}(\tau \ast \psi) = \bigcap_{a < \varphi+\omega} D_a(\tau \ast \psi) = \bigcap_{n=0}^{\infty} D_{\varphi+2n}(\tau \ast \psi)
\subset \bigcap_{n=0}^{\infty} D_{\varphi+n}(\tau) = D_{\varphi+\omega}(\tau).
\]

In an analogous way we prove that \( D_{\varphi+\omega}(\tau \ast \psi) \subset D_{\varphi+\omega}(\psi) \), thus

\[
D_{\varphi+\omega}(\tau \ast \psi) \subset D_{\varphi+\omega}(\tau) \cap D_{\varphi+\omega}(\psi).
\]
Now taking into account again the intersection definition of the sets $D_\alpha$ for limit numbers $\alpha$, and again using the induction we get the required conclusion (4).

If $f: X \to Y$ is a surjection between compact spaces $X$ and $Y$, and $\tau = \{ X_\alpha \}_{\alpha < \Gamma}$ is a covering sequence of $X$, then the covering sequence $\{ f(X_\alpha) \}_{\alpha < \Gamma}$ of $Y$ will be denoted by $f(\tau)$. The proof of the following is quite similar to the proof of Lemma 17(a) of [P1], so it is omitted.

If $\tau$ is a covering sequence of a compact space $X$ and $f$:

\[(5) \quad X \to Y \text{ is a surjection, then} \]

\[D_\alpha(f(\tau)) \subseteq f(D_\alpha(\tau)) \quad \text{for every } \alpha < \Gamma.\]

Let a class $\mathcal{F}$ of compact spaces be invariant with respect to continuous mappings, i.e., we assume that each continuous image of any member of $\mathcal{F}$ belongs to $\mathcal{F}$. Denote by $\mathcal{F}_1$ the class of all compact spaces being countable unions of their compact subsets belonging to $\mathcal{F}$. Then, of course, each element of $\mathcal{F}_1$ admits covering sequences composed of elements of $\mathcal{F}$. Let $X \in \mathcal{F}_1$. Fix a limit ordinal $\Gamma$ with cardinality greater than $\text{card} \ X$. Further, put

\[S(X) = \{ \tau: \tau = \{ X_\alpha \}_{\alpha < \Gamma} \text{ is a covering sequence of } X \text{ with} \]

\[X_\alpha \in \mathcal{F} \text{ for every } \alpha < \Gamma \}.\]

Then for every $\alpha < \Gamma$ we put

\[W_\alpha(X, \mathcal{F}) = \bigcap \{ D_\alpha(\tau): \tau \in S(X) \}.\]

We need the following two properties of these sets.

There is the greatest limit number $\varphi_0 < \Gamma$ such that

\[(6) \quad W_{\varphi_0}(X, \mathcal{F}) \neq \emptyset.\]

In fact, the sets $W_\alpha(X, \mathcal{F})$ form a decreasing sequence of compact subsets of $X$ with some $W_\alpha(X, \mathcal{F}) = \emptyset$ (see (3)). By the definitions we have $W_\varphi(X, \mathcal{F}) = \bigcap\{ W_\alpha(X, \mathcal{F}): \alpha < \varphi \}$ for limit ordinals $\varphi$. Thus the number $\alpha_0 = \min\{ \alpha: W_\alpha(X, \mathcal{F}) = \emptyset \}$ is nonlimit, so $\alpha_0 = \varphi_0 + n$, where $\varphi_0$ is limit and $n > 0$ is natural. The number $\varphi_0$ satisfies the required condition.

There is a covering sequence $\tau \in S(X)$ with $\lambda(\tau) = \varphi_0 + n$

\[(7) \quad \text{for some natural } n > 0, \text{ where } \varphi_0 \text{ is as in (6).}\]

Indeed, since $\emptyset = W_{\varphi_0 + \omega}(X, \mathcal{F}) = \bigcap\{ D_{\varphi_0 + \omega}(\tau): \tau \in S(X) \}$ and since the sets $D_\alpha(\tau)$ are compact, there is a finite system $\tau_1, \ldots, \tau_m \in S(X)$ with $\bigcap\{ D_{\varphi_0 + \omega}(\tau_i): \tau \in S(X) \}$ and some $\varphi_0 + n$. Put $\tau = (\cdots (\tau_1 \ast \tau_2) \ast \cdots \ast \tau_m)$. Then $D_{\varphi_0 + \omega}(\tau) \subseteq \bigcap\{ D_{\varphi_0 + \omega}(\tau_i): \tau \in S(X) \}$ and $\lambda(\tau) = \varphi_0 + n$ for some natural $n > 0$.

2. Lemma. If $X \in \mathcal{F}_1$ and $f: X \to Y$ is a surjection, then for each limit ordinal $\varphi < \Gamma$ we have

\[W_\varphi(Y, \mathcal{F}) \subseteq f(W_\varphi(X, \mathcal{F})).\]

Proof. Let $y \in f(W_\varphi(X, \mathcal{F}))$, i.e.,

\[\emptyset = f^{-1}(y) \cap W_\varphi(X, \mathcal{F}) = f^{-1}(y) \cap \bigcap \{ D_\varphi(\tau): \tau \in S(X) \}
\]

\[= \bigcap \{ f^{-1}(y) \cap D_\varphi(\tau): \tau \in S(X) \}.\]
Since all of the sets \( f^{-1}(y) \cap D_{\phi}(\tau) \) are closed subsets of the compact space \( X \), there is a finite system \( \tau_1, \ldots, \tau_m \in S(X) \) such that \( \cap \{ f^{-1}(y) \cap D_{\phi}(\tau_i) : i \in \{1, \ldots, m\} \} = \emptyset \). Put \( \tau = (\cdots (\tau_1 \ast \tau_2) \ast \cdots \ast \tau_{m-1}) \ast \tau_m \). Then \( \tau \in S(X) \), and by (4) we get
\[
 f^{-1}(y) \cap D_{\phi}(\tau) \subseteq f^{-1}(y) \cap D_{\phi}(\tau_1) \cap \cdots \cap D_{\phi}(\tau_m) = \emptyset,
\]
hence \( y \not\in f(D_{\phi}(\tau)) \). By (5) and by the definition of \( W_{\phi}(Y, \mathcal{F}) \), noting that \( f(\tau) \in S(Y) \), we see \( W_{\phi}(Y, \mathcal{F}) \subseteq D_{\phi}(f(\tau)) \subseteq f(D_{\phi}(\tau)) \). Thus \( y \not\in W_{\phi}(Y, \mathcal{F}) \), which completes the proof.

The following lemma plays a crucial role in the proof of Theorem 1.

3. Lemma. If \( C_1, C_2, \ldots \) is a countable sequence of compact subsets of a compact space \( X \) such that \( X = C_1 \cup C_2 \cup \cdots \), then there is a compact nonempty subset \( F \) of \( X \) such that

(1) \( F \subset f(F) \) for every surjection \( f : X \to X \), and
(2) there are a finite sequence \( C_{i_1}, \ldots, C_{i_n} \) of sets and a finite sequence of mappings \( f_{i_k} : C_{i_k} \to X \) for \( k \in \{1, \ldots, n\} \), such that \( F \subset f_{i_1}(C_{i_1}) \cup \cdots \cup f_{i_n}(C_{i_n}) \).

Proof. Let \( \mathcal{F} \) be the class of all continuous images of the sets \( C_1, C_2, \ldots \). Then, of course, \( \mathcal{F} \) is invariant with respect to continuous mappings and \( X \in \mathcal{F} \). Put \( F = W_{\phi_0}(X, \mathcal{F}) \), where \( \phi_0 \) is the number guaranteed by (6). Then by Lemma 2 we get \( F \subset f(F) \) for each surjection \( f : X \to X \). Let \( \tau = \{ X_\alpha \}_{\alpha < 1} \) be a covering sequence guaranteed by (7). Then since \( D_{\phi_0+\alpha}(\tau) = \emptyset \) we have
\[
 F = W_{\phi_0}(X, \mathcal{F}) \subseteq D_{\phi_0}(\tau) \subseteq X_{\phi_0} \cup X_{\phi_0+1} \cup \cdots \cup X_{\phi_0+(n-1)},
\]
and \( X_{\phi_0+i} \in \mathcal{F} \) for every \( i \in \{0, 1, \ldots, n-1\} \), thus the sets \( X_{\phi_0+i} \) are continuous images of some sets \( C_{i_k} \), which completes the proof.

Proof of Theorem 1. Let \( F \subset X \) be a set guaranteed by Lemma 3 for \( C_i = \text{cl} A_i \). By (2) of Lemma 3 the set \( F \) may be covered by finitely many subcontinua of each containing a dense \( \mathcal{K} \)-connected subset. Let \( F_1, \ldots, F_m \) be such sets with an additional assumption that \( m \) is the minimum number. Without loss of generality we may assume that \( F_i = \text{cl} B_i \) for some \( \mathcal{K} \)-components \( B_i \) of \( X \) for \( i \in \{1, \ldots, m\} \).

Let \( \mathcal{A} \) be the family of all \( \mathcal{K} \)-components of \( X \).

(8) For every \( A \in \mathcal{A} \) we have \( \text{cl} A \cap F \neq \emptyset \).

In fact, consider any surjection \( f : X \to X \) sending a point of \( B_1 \) to a point of \( A \). Then \( f(B_1) \subset A \) and \( f(F_1) = f(\text{cl} B_1) \subset \text{cl} A \). If \( \text{cl} A \cap F \) were empty, the union \( f(F_2) \cup \cdots \cup f(F_m) \) of \( m-1 \) continua with dense \( \mathcal{K} \)-connected subsets would contain \( F \) (since \( F \subset f(F) \subset f(F_1) \cup \cdots \cup f(F_m) \) and \( f(F_1) \cap F = \text{cl} A \cap F = \emptyset \)), contrary to the assumption on \( m \).

Put \( \mathcal{A}_i = \{ A \in \mathcal{A} : \text{cl} A \cap F_i \neq \emptyset \} \) and \( G_i = \text{cl}(\bigcup \mathcal{A}_i) \) for \( i \in \{1, \ldots, m\} \). We prove that
\[
 G_i \cap G_j = \emptyset \quad \text{for } i \neq j, \ i, j \in \{1, \ldots, m\}.
\]

Suppose \( x \in G_i \cap G_j \) with \( j > i \). Let \( U \) be the nonempty interior of a \( \mathcal{K} \)-component \( B_{m+1} \) of \( X \) and let \( f : X \to X \) be a surjection sending \( x \) to a point \( y \in U \). Thus \( f(A), f(B) \subset B_{m+1} \) for some \( A \in \mathcal{A}_i \) and \( B \in \mathcal{A}_j \), and \( \text{cl} B_{m+1} \cap f(F_i) \neq \emptyset \neq \text{cl} B_{m+1} \cap f(F_j) \). Let a surjection \( g : X \to X \) send a point \( p \in \text{cl} B_{m+1} \cap f(F_i) \) to \( y \).
Then \( g(B_{m+1}) \subseteq B_{m+1} \) and \( \text{cl} B_{m+1} \cap g(f) \neq \emptyset \). Let a surjection \( h \colon X \to X \) send a point \( q \in \text{cl} B_{m+1} \cap g(f) \) to \( y \). Then \( h(B_{m+1}) \), \( hgf(B_j) \subseteq B_{m+1} \). Therefore

\[
F \subseteq \text{hgf}(F) \subseteq \text{hgf}(F_1) \cup \cdots \cup \text{hgf}(F_m)
\]

\[
\subseteq \text{hgf}(F_1) \cup \cdots \cup \text{hgf}(F_{i-1}) \cup \text{hgf}(F_{i+1}) \cup \cdots \cup \text{hgf}(F_{j-1})
\]

\[
\cup \text{hgf}(F_{j+1}) \cup \cdots \cup \text{hgf}(F_m) \cup \text{cl} B_{m+1}.
\]

Thus \( m - 1 \) sets with dense \( \mathcal{K} \)-connected subsets cover \( F \), contrary to the assumption on \( m \).

The statement (9), by (8) and by the connectedness of \( X \), implies

(10) \( m = 1 \).

(11) For every \( A \in \mathcal{A} \) we have \( F \subseteq \text{cl} A \).

For, let a surjection \( g \colon X \to X \) send a point of \( B_1 \) to a point of \( A \). Then \( F \subseteq g(F) \subseteq g(\text{cl} B_1) = \text{cl} g(B_1) \subseteq \text{cl} A \).

(12) For every \( A \in \mathcal{A} \) we have \( \text{cl} A = X \).

Indeed, for a given point \( x \in X \) let a surjection \( f \colon X \to X \) send a point \( y \in F \) to \( x \), and let \( B \in \mathcal{A} \) be a \( \mathcal{K} \)-component of \( X \) such that \( f(B) \subseteq A \). Then \( y \in \text{cl} B \) by (11). Therefore \( x = f(y) \in f(\text{cl} B) = \text{cl} f(B) \subseteq \text{cl} A \). Thus we have (12).

To make the proof of Theorem 1 complete, note that since every \( \mathcal{K} \)-component of \( X \) is dense in \( X \) and one \( \mathcal{K} \)-component has nonempty interior, this \( \mathcal{K} \)-component is the only one.

Applications and questions. As an immediate consequence of Theorem 1 and of the Baire theorem we have the following corollary.

4. Corollary. If a continuously homogeneous continuum is the countable union of arcwise connected (\( \mathcal{K} \)-connected) continua, then it is arcwise connected (\( \mathcal{K} \)-connected).

5. Theorem. If a continuously homogeneous nondegenerate metric continuum \( X \) contains no simple triod and it has only countably many arc components, then \( X \) is either an arc or a simple closed curve.

Proof. Let \( A \) be an arc component of \( X \). Then one of the following statements is true:

(1) \( A \) is degenerate,
(2) \( A \) is a simple closed curve,
(3) \( A \) is nondegenerate and it contains no simple closed curve.

In fact, note that if \( A \) contains a simple closed curve, then \( A \) itself is a simple closed curve (otherwise \( A \) would contain a simple triod).

In case (3), since \( A \) contains no simple closed curve, for all points \( a, b \in A \) with \( a \neq b \) there is only one arc \( ab \) in \( A \). Further we observe that in this case one of the following is true:

(3.1) \( A \) has two end points,
(3.2) \( A \) has one end point,
(3.3) \( A \) has no end point.
Namely, if \( e_1, e_2 \in A \) are distinct end points of \( A \), then each point \( p \in A \) belongs to \( e_1 e_2 \). Indeed, if not, let \( q \) be the first point of the arc \( p e_1 \) lying in the arc \( e_1 e_2 \). Then \( pq \cup q e_1 \cup q e_2 \) is a simple triod for \( q \neq e_1 \) and \( q \neq e_2 \). Thus (3.1)–(3.3) are all possibilities and in case (3.1) \( A \) is an arc.

In case (3.2) let \( e \) be the end point of \( A \). Then we inductively construct a well-ordered sequence \( (A_\alpha) \) of arcs contained in \( A \) with \( \bigcup_\alpha A_\alpha = A \). Namely,
\[
A_0 = ep \quad \text{for a point} \quad p \in A \setminus \{ e \},
A_\alpha = eq \quad \text{for a point} \quad q \in A \setminus \bigcup \{ A_\beta : \beta < \alpha \}, \quad \text{for} \quad \alpha > 0
\]
(if such \( q \) exists). Let \( A_\alpha = ex \) and \( A_\beta = ey \) for \( \alpha > \beta \). We have \( x \notin ey \), thus \( ex \not\subset ey \), so \( ey \not\subseteq ex \). Hence the sequence \( \{ A_\alpha \} \) is increasing, thus countable.

Further, we may observe that it is a one-to-one image of the half-line.

Similarly we prove that in case (3.3) \( A \) is a one-to-one image of the real line (details of this proof are left to the reader).

Each of these cases implies that \( A \) is an \( F_\alpha \)-set, thus each arc component of \( X \) is an \( F_\alpha \)-set. Since \( X \) has countably many arc components only, by the Baire theorem, we infer that at least one of them has nonempty interior. Applying Theorem 1 we see that \( X \) is arcwise connected. Thus \( X \) is the only arc component satisfying either (2) or (3). Suppose it is neither an arc nor a simple closed curve. Thus it is nonlocally connected (since nondegenerate atriodic locally connected continuum is either an arc or a simple closed curve). There are two possibilities only: (3.2) and (3.3), i.e., \( X \) is a one-to-one image either of the half-line or of the real line. But Krupski showed in [K1, Theorem 4, p. 352] that compact nonlocally connected one-to-one images of the half-line or of the real-line are not continuously homogeneous. This contradiction completes the proof.

J. J. Charatonik and T. Maćkowiak posed in [CM, Problem 3.11] the problem of characterizing continuously homogeneous arc-like continua. The former of the following two corollaries may be considered as a step in a way to do it. It also improves Corollary 1 of [K1, p. 354].

6. Corollary. Let \( X \) be a nondegenerate continuously homogeneous metric continuum with only countably many arc components. Then the following statements are equivalent:

(a) \( X \) is an arc,
(b) \( X \) is arc-like,
(c) \( X \) contains neither a simple triod nor a simple closed curve.

7. Corollary. Under the same assumptions as in Corollary 6, the following statements are equivalent:

(a) \( X \) is a simple closed curve,
(b) \( X \) is circle-like,
(c) \( X \) is not an arc and it contains no simple triod.

There are some interesting questions closely related to the subject of this paper, and also to the results of [K1, K2, CG].
QUESTION 1. If a continuum $X$ is continuously homogeneous and has $\mathcal{X}$-components (arc components) $A_1, A_2, \ldots$ with $X = \text{cl } A_1 \cup \text{cl } A_2 \cup \cdots$, is each $\mathcal{X}$-component (arc component) of $X$ necessarily dense in $X$?

We know that each arc component of a continuously homogeneous continuum $X$ with finitely many arc components is dense in $X$ (see Theorem 3 of [CG]).

QUESTION 2. If a continuously homogeneous continuum $X$ has only countably many $\mathcal{X}$-components (arc components), is each $\mathcal{X}$-component (arc component) necessarily dense in $X$?

QUESTION 3. Under the same conditions as in Question 2, is $X$ necessarily $\mathcal{X}$-connected (arcwise connected)?

QUESTION 4. What about an answer to Question 3 if we additionally assume that $X$ has only a finite number of $\mathcal{X}$-components (arc components)?

REFERENCES


[K1] P. Krupski, Continua which are homogeneous with respect to continuity, Houston J. Math. 5 (1979), 345–356.


[P2] ____, Some invariants under perfect mappings and their applications to continua (to appear).

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