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HOMOCLINIC INTERSECTIONS AND INDECOMPOSABILITY

MARCY BARGE

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ABSTRACT. The closure of a one-dimensional unstable manifold of a hyperbolic fixed point of a diffeomorphism having homoclinic points is, under mild assumptions, shown to be an indecomposable continuum. As a result, dynamical systems possessing such behavior cannot be modeled using inverse limits based on any simple space.

I. Introduction. In [**W**] R. F. Williams demonstrated that every hyperbolic, one-dimensional, expanding attractor for a discrete dynamical system is topologically conjugate to the induced map on an inverse limit space based on a branched one-manifold. Examples [**B**] show that some nontrivial one-dimensinal nonhyperbolic attractors are also conjugate to inverse limit systems over branched one-manifolds.

In a complicated attractor (or invariant set) one expects to see periodic points possessing homoclinic orbits. If this is the case in a nondegenerate way, our Theorem provides the existence of a topologically indecomposable invariant set for the dynamical system. As a corollary we find that in certain commonly occurring situations (at certain parameter values in the Hénon map, for example) there are one-dimensional attractors (or invariant sets) that are not conjugate to any inverse limit system over a branched one-manifold.

II. Let $F\colon M \to M$ be a C^1 diffeomorphism of the m-manifold M and let $p \in M$ be a hyperbolic fixed point of F with stable manifold $W^s(p)$ and one-dimensional unstable manifold $W^u(p)$. Let $W^{u+}(p)$ be one of the branches of $W^u(p)$ and assume that $F(W^{u+}(p)) = W^{u+}(p)$ (otherwise, replace F by F^2).

By Hartman's Theorem there is a homeomorphism

$$\psi\colon B_1 \to M, \quad B_1 = \left\{ (x_1, \ldots, x_m) \in \mathbf{R}^m \middle| \sum_{i=1}^{m-1} x_i^2 \leq 1, 0 \leq x_m \leq 1 \right\},$$

such that

$$\psi\left(\left\{(x_1, \ldots, x_{m-1}, 0) \middle| \sum_{i=1}^{m-1} x_i^2 \leq 1\right\}\right) = W^s_{\mathrm{loc}}(p),$$

$$\psi\left(\{(0, \ldots, 0, x_m) | 0 \leq x_m \leq 1\}\right) = W^{u+}_{\mathrm{loc}}(p)$$

(the local stable and unstable manifolds of p, respectively) and $A = \psi^{-1} \circ F \circ \psi$ is linear where defined.

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We consider the following conditions on F:

(A1) $\mathrm{cl}(W^{u+}(p)) =$ closure of $w^{u+}(p)$ is compact.

(A2) There is an arc α in $\psi(B_1) \cap W^{u+}(p)$ such that $\alpha \cap W^s_{\mathrm{loc}}(p) \neq \varnothing$ and $\alpha \not\subseteq W^s_{\mathrm{loc}}(p)$.

(A3) There is an essential $m-1$ sphere, S^{m-1}, in $W^s_{\mathrm{loc}}(p) - \{p\}$ such that $S^{m-1} \cap \mathrm{cl}(W^{u+}(p)) = \varnothing$.

By a continuum we will mean a compact and connected metric space. A continuum is said to be indecomposable if it is not the union of two proper subcontinua. Equivalently, a continuum is indecomposable if every proper subcontinuum has empty interior.

THEOREM. *Suppose that $F\colon M \to M$ is a C^1 diffeomorphism with hyperbolic fixed point p as above that satisfies* (A1)–(A3). *Then $\mathrm{cl}(W^{u+}(p))$ is an indecomposable continuum.*

PROOF. Let $C = \{(x_1, \dots, x_m) | (x_1, \dots, x_{m-1}, 0) \in \psi^{-1}(S^{m-1}), 0 \leq x_n \leq \varepsilon\}$ where $\varepsilon > 0$ is small enough so that $\psi(C) \cap \mathrm{cl}(W^{u+}(p)) = \varnothing$. For $n \geq 0$, let $C_n = A^n(C) \cap B_1 = \psi^{-1}(\mathrm{comp}(F^n(\psi(C))))$ where $\mathrm{comp}(F^n(\psi(C)))$ denotes the connected component of $F^n(\psi(C)) \cap \psi\ (B_1)$ containing $F^n(S^{m-1})$. Note that $\psi(C_n) \cap \mathrm{cl}(W^{u+}(p)) = \varnothing$ for all n since $\mathrm{cl}(W^{u+}(p))$ is invariant under F. The hyperbolicity of F at p guarantees that, given any r, $0 < r \leq 1$, there is an $n = n(r)$ large enough so that C_n separates $\{(0, \dots, 0, x_m) | 0 \leq x_m \leq 1\}$ from $\left\{(x_1, \dots, x_{m-1}, 0) | \sum_{i=1}^{m-1} x_i^2 = r\right\}$ in B_1.

Let $G\colon \mathbf{R}^+ \cap \{0\} \to W^{u+}(p)$ be a continuous, one-to-one, onto parameterization of $W^{u+}(p)$. We claim that $\mathrm{cl}(G([y, \infty))) = \mathrm{cl}(W^{u+}(p))$ for any $y \in \mathbf{R}^+$. Indeed, let $\alpha = G([x, z])$ with $G(x) \in W^s_{\mathrm{loc}}(p)$ be as in (A2) and, for $n \geq 0$, let α_n be the connected component of $F^n(\alpha) \cap \psi(B_1)$ containing $F^n(G(x))$. Hyperbolicity of F at p implies that $\limsup \alpha_n = W^{u+}_{\mathrm{loc}}(p)$. For all sufficiently large n, $\alpha_n \subseteq G([y, \infty))$ so that $\mathrm{cl}(G([y, \infty))) \supseteq W^u_{\mathrm{loc}}(p)$. Thus

$$\begin{aligned}
\mathrm{cl}(W^{u+}(p)) &= \mathrm{cl}\left(C \bigcup_{n \geq 0} F^n(W^{u+}_{\mathrm{loc}}(p))\right) \\
&\subseteq \mathrm{cl}\left(\bigcup_{n \geq 0} F^n(\mathrm{cl}(G([y, \infty))))\right) \\
&= \mathrm{cl}\left(\bigcup_{n \geq 0} \mathrm{cl}(F^n(G([y, \infty))))\right) \\
&\subseteq \mathrm{cl}(G([y, \infty))).
\end{aligned}$$

Now suppose that $\mathrm{cl}(W^{u+}(p))$ is decomposable. Let $H \subseteq \mathrm{cl}(W^{u+}(p))$ be a proper subcontinuum such that interior(H), relative to $\mathrm{cl}(W^{u+}(p))$, is nonempty. Then $H \cap W^{u+}(p) \neq \varnothing$ and, since H is proper, we have from the preceding paragraph that for no $y \geq 0$ is $G([y, \infty))$ contained in H. We also have (from the preceding paragraph) that interior$(G([y_1, y_z]))$, relative to $\mathrm{cl}(W^{u+}(p))$, is empty for all $0 \leq y_1 < y_2 < \infty$. Thus there are numbers $0 \leq y_1 < y_2 < y_3 < y_4$ such that $G(y_1), G(y_3) \notin H$ and $G(y_2), G(y_4) \in H$. Now let N be large enough so that

$F^{-N}(G(y_i)) \in W^{u+}_{\text{loc}}(p)$ for $i = 1,2,3,4$ and let $0 \le x^1_m < x^2_m < x^3_m < x^4_m \le 1$ be such that $\psi((0,\dots,x^i_m)) = F^{-N}(G(y_i))$ for $i = 1,2,3,4$. Since $F^{-N}(H)$ is closed in M and $F^{-N}(G(y_1)), F^{-N}(G(y_3)) \notin F^{-N}(H)$, there is an r, $0 < r \le 1$, such that

$$\psi\left(\left\{(x_1,\dots,x_{m-1},x^i_m) \mid \sum_{j=1}^{m-1} x_j^2 \le r\right\}\right) \cap F^{-N}(H) = \varnothing$$

for $i = 1$ and $i = 3$.

Given r as above, let $n = n(r)$ be as in the first paragraph of this proof. Let $D = \{(x_1,\dots,x_m) \mid (x_1,\dots,x_m) \in C_n$ and $x^1_m \le x^2_m \le x^3_m$, or $\sum_{j=1}^{m-1} x_j^2 \le r$ and $x_m = x^1_m$, or $\sum_{j=1}^{m-1} x_j^2 \le r$ and $x_m = x^3_m\}$.

Then $\psi(D) \cap F^{-N}(H) = \varnothing$ and $\psi(D)$ separates $f^{-N}(G(y_2))$ from $F^{-N}(G(y_4))$ in M. But then $F^{-N}(H)$, and hence H itself, is not connected.

Thus, no proper subcontinuum of $\text{cl}(W^{u+}(p))$ has nonempty interior and $\text{cl}(W^{u+}(p))$ is indecomposable.

III. In this section we apply our theorem to obtain the corollary mentioned in the Introduction.

Given a continuum X and a point $x \in X$, the composant determined by x, C_x, is the union of all proper subcontinua of X that contain x. If X is indecomposable, the composants of X partition X, there are uncountably many distinct composants, and each is dense in X (see [**HY**]).

By a topological branched one-manifold we will mean a compact, connected metric space that is locally homeomorphic to a one-point union of finitely many arcs (open, closed, or half-open, half-closed). Given a branched one-manifold K with metric d and a continuous map $f\colon K \to K$, the inverse limit space (K,f) is the space $(K,f) = \{\mathbf{x} = (x_0,x_1,\dots) \mid x_n \in K$ and $f(x_{n+1}) = x_n$ for all $n \ge 0\}$ with metric $\mathbf{d}$ given by

$$\mathbf{d}(\mathbf{x},\mathbf{y}) = \sum_{n=0}^{\infty} \frac{d(x_n,y_n)}{2n}.$$

(K,f) is a continuum and the induced map on (K,f) is the homeomorphism (onto) $\hat{f}\colon (K,f) \to (K,f)$ given by $\hat{f}((x_0,x_1,\dots)) = (f(x_0),x_0,\dots)$.

Maps $f\colon X \to X$ and $g\colon Y \to Y$ of the topological spaces X and Y are said to be topologically conjugate if there is a homeomorphism h from X onto Y such that $h^{-1} \circ g \circ h = f$.

In the corollary, we have in mind the dynamical situation pictured in Figure 1.

COROLLARY. *Let F be as in the theorem and in addition assume that* $\text{cl}(W^{u+}(p)) \cap W^s(p) \subseteq W^{w+}(p)$. *Then for no continuous map f of a branched one-manifold K is $F|_{\text{cl}(W^{u+}(p))}$ topologically conjugate to the induced map $\hat{f}$.*

PROOF. Suppose there is a branched one-manifold K, a continuous map $f\colon K \to K$, and a homeomorphism h from $\text{cl}(W^{u+}(p))$ onto (K,f) such that $h^{-1} \circ f \circ h = F|_{\text{cl}(W^{u+}(p))}$. Without loss of generality, we may assume that f is onto, otherwise replace K by $L = \bigcap_{n\ge 0} f^n(L)$ ($\hat{f}\colon (K,f) \to (K,f)$ and $\hat{f}|_L = (L, f|_L)$ are topologically conjugate).

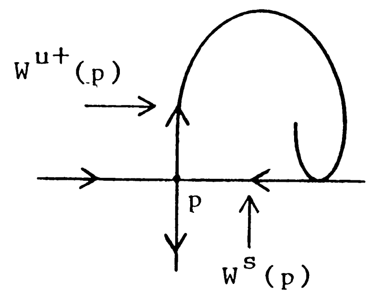

FIGURE 1

Since $W^{u+}(p) = \bigcup_{n\geq 0} G([0,n])$ we see that the composant of $\mathrm{cl}(W^{u+}(p))$ determined by p in $\mathrm{cl}(W^{u+}(p)), C_p$, contains $W^{u+}(p)$. Thus, the additional assumption in this corollary means that the stable set of p in $\mathrm{cl}(W^{u+}(p))$, that is $\{x \in W^{u+}(p) | F^n(x) \to p \text{ as } n \to \infty\}$, is contained in C_p.

Thus, $S(h(p)) = \{\mathbf{x} \in (K,f) | \hat{f}(\mathbf{x}) \to h(p) \text{ as } n \to \infty\}$ must be contained in $C_{h(p)}$, the composant determined by $h(p)$ in (K,f).

Since h conjugates $\hat{f}$ and $F|_{\mathrm{cl}(W^{u+}(p))}$ and p is a fixed point of G, $h(p)$ must be fixed by $\hat{f}$ so that $h(p) = (p_0, p_0, \dots)$ for some $p_0 \in K$.

It is clear from the definition of a branched one-manifold that there is an $l < \infty$ (depending of K) such that any collection of l distinct points in K separates K. Now let $\mathbf{x} = (x_0, x_1, \dots) \in S(h(p)) - \{h(p)\}$. Then $\mathbf{x}, \hat{f}(\mathbf{x}), \dots, \hat{f}^{l-1}(\mathbf{x})$ are l distinct points in $S(h(p))$. It follows that there is an $N \geq 0$ such that $\pi_N(\hat{f}^i(\mathbf{x})) \neq \pi(\hat{f}^j(\mathbf{x}))$ for $0 \leq i < j \leq l-1$ where π_N is projection onto the Nth coordinate. Thus $K - \{\pi_N(\mathbf{x}), \dots, \pi_N(f^{l-1}(\mathbf{x}))\} = K - \{x_N, f(x_N), \dots, f^{l-1}(x_N)\}$ is not connected.

By the theorem, (K,f) is indecomposable. Thus there is a composant C in (K,f) such that $C \cap C_{h(p)} = \varnothing$. In addition, C is dense in (K,f) and is connected. π_N is continuous so that $\pi_N(C)$ is connected and dense in K. It must then be the case that $\pi_N(C) \cap \{x_n, \dots, f^{l-1}(x_N)\} \neq \varnothing$. Say $\mathbf{y} = (y_0, y_1, \dots) \in C$ and i, $0 \leq i \leq l-1$, are such that $\pi_N(\mathbf{y}) - y_N = f^i(x_N)$.

Now, since $\mathbf{x} \in S(h(p))$, $\lim_{n\to\infty} f^n(x_0) = p_0$. Thus,

$$\lim_{n\to\infty} f^n(y_0) = \lim_{n\to\infty} f^{n+N}(f^i(x_N)) = \lim_{n\to\infty} f^{n+i}(x_0) = p_0$$

and it follows that $\mathbf{y} \in S(h(p))$. But this is impossible since $\mathbf{y} \in C$, $C \cap C_{h(p)} = \varnothing$ and $S(h(p)) \subseteq C_{h(p)}$. Thus, there can be no such K, f, and h.

REFERENCES

[**B**] M. Barge, *A method for constructing attractors*, preprint.

[**HY**] J. Hocking and G. Young, *Topology*, Addison-Wesley, 1961.

[**W**] R. F. Williams, *One-dimensional non-wandering sets*, Topology **6** (1967), 473–483.

DEPARTMENT OF MATHEMATICAL SCIENCES, MONTANA STATE UNIVERSITY, BOZEMAN, MONTANA 59717-0001