HOMOCLINIC INTERSECTIONS AND INDECOMPOSABILITY

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ABSTRACT. The closure of a one-dimensional unstable manifold of a hyperbolic fixed point of a diffeomorphism having homoclinic points is, under mild assumptions, shown to be an indecomposable continuum. As a result, dynamical systems possessing such behavior cannot be modeled using inverse limits based on any simple space.

I. Introduction. In [W] R. F. Williams demonstrated that every hyperbolic, one-dimensional, expanding attractor for a discrete dynamical system is topologically conjugate to the induced map on an inverse limit space based on a branched one-manifold. Examples [B] show that some nontrivial one-dimensional nonhyperbolic attractors are also conjugate to inverse limit systems over branched one-manifolds.

In a complicated attractor (or invariant set) one expects to see periodic points possessing homoclinic orbits. If this is the case in a nondegenerate way, our Theorem provides the existence of a topologically indecomposable invariant set for the dynamical system. As a corollary we find that in certain commonly occurring situations (at certain parameter values in the Hénon map, for example) there are one-dimensional attractors (or invariant sets) that are not conjugate to any inverse limit system over a branched one-manifold.

II. Let $F: M \to M$ be a $C^1$ diffeomorphism of the $m$-manifold $M$ and let $p \in M$ be a hyperbolic fixed point of $F$ with stable manifold $W^s(p)$ and one-dimensional unstable manifold $W^u(p)$. Let $W^u+(p)$ be one of the branches of $W^u(p)$ and assume that $F(W^u+(p)) = W^u+(p)$ (otherwise, replace $F$ by $F^2$).

By Hartman’s Theorem there is a homeomorphism

$$
\psi: B_1 \to M, \quad B_1 = \left\{ (x_1, \ldots, x_m) \in \mathbb{R}^m \middle| \sum_{i=1}^{m-1} x_i^2 \leq 1, 0 \leq x_m \leq 1 \right\},
$$

such that

$$
\psi \left( \left\{ (x_1, \ldots, x_{m-1}, 0) \middle| \sum_{i=1}^{m-1} x_i^2 \leq 1 \right\} \right) = W^s_{loc}(p),
$$

$$
\psi \left( \left\{ (0, \ldots, 0, x_m) \right\} \right) = W^u_{loc}(p)
$$

(the local stable and unstable manifolds of $p$, respectively) and $A = \psi^{-1} \circ F \circ \psi$ is linear where defined.
We consider the following conditions on $F$:

(A1) $\text{cl}(W_u^+(p)) = \text{closure of } w_u^+(p)$ is compact.

(A2) There is an arc $\alpha$ in $\psi(B_1) \cap W_u^+(p)$ such that $\alpha \cap W_{\text{loc}}^s(p) \neq \emptyset$ and $\alpha \not\subset W_{\text{loc}}^s(p)$.

(A3) There is an essential $m-1$ sphere, $S^{m-1}$, in $W_{\text{loc}}^s(p) - \{p\}$ such that $S^{m-1} \cap \text{cl}(W_u^+(p)) = \emptyset$.

By a continuum we will mean a compact and connected metric space. A continuum is said to be indecomposable if it is not the union of two proper subcontinua. Equivalently, a continuum is indecomposable if every proper subcontinuum has empty interior.

**THEOREM.** Suppose that $F: M \to M$ is a $C^1$ diffeomorphism with hyperbolic fixed point $p$ as above that satisfies (A1)-(A3). Then $\text{cl}(W_u^+(p))$ is an indecomposable continuum.

**PROOF.** Let $C = \{(x_1, \ldots, x_m) | (x_1, \ldots, x_{m-1}, 0) \in \psi^{-1}(S^{m-1}), 0 < x_n \leq \varepsilon\}$ where $\varepsilon > 0$ is small enough so that $\psi(C) \subset W_u^+(p)$ is in the same connected component as $p$. For $n \geq 0$, let $C_n = A^n(C) \cap B_1 = \psi^{-1}(\text{comp}(F^n(\psi(C))))$ where $\text{comp}(F^n(\psi(C)))$ denotes the connected component of $F^n(\psi(C)) \cap \psi(B_1)$ containing $F^n(S^{m-1})$. Note that $\psi(C_n) \subset \text{cl}(W_u^+(p))$ is a proper subcontinuum. The hyperbolicity of $F$ at $p$ guarantees that, given any $r$, $0 < r \leq 1$, there is an $n = n(r)$ large enough so that $C_n$ separates $\{(0, \ldots, x_m)|0 \leq x_m \leq 1\}$ from $\{(x_1, \ldots, x_{m-1}, 0)|\sum_{i=1}^{m-1} x_i^2 = r\}$ in $B_1$.

Let $G: \mathbb{R}^+ \cap \{0\} \to W_u^+(p)$ be a continuous, one-to-one, onto parameterization of $W_u^+(p)$. We claim that $\text{cl}(G([y, \infty))) = \text{cl}(W_u^+(p))$ for any $y \in \mathbb{R}^+$. Indeed, let $\alpha = G([-x, z])$ with $G(x) \in W_{\text{loc}}^s(p)$ be as in (A2) and, for $n \geq 0$, let $\alpha_n$ be the connected component of $F^n(\alpha) \cap \psi(B_1)$ containing $F^n(G(x))$. Hyperbolicity of $F$ at $p$ implies that $\alpha_n = W_{\text{loc}}^s(p)$. For all sufficiently large $n$, $\alpha_n \subset G([y, \infty))$ so that $\text{cl}(G([y, \infty])) \supset W_u^+(p)$. Thus

$$\text{cl}(W_u^+(p)) = \text{cl} \left( \bigcup_{n \geq 0} F^n(W_{\text{loc}}^s(p)) \right) \subset \text{cl} \left( \bigcup_{n \geq 0} F^n(\text{cl}(G([y, \infty)))) \right) = \text{cl} \left( \bigcup_{n \geq 0} \text{cl}(F^n(G([y, \infty]))) \right) \subset \text{cl}(G([y, \infty]))).$$

Now suppose that $\text{cl}(W_u^+(p))$ is decomposable. Let $H \subset \text{cl}(W_u^+(p))$ be a proper subcontinuum such that interior($H$), relative to $\text{cl}(W_u^+(p))$, is nonempty. Then $H \cap W_u^+(p) \neq \emptyset$ and, since $H$ is proper, we have from the preceding paragraph that for no $y \geq 0$ is $G([y, \infty))$ contained in $H$. We also have (from the preceding paragraph) that interior($G([y_1, y_2])$), relative to $\text{cl}(W_u^+(p))$, is empty for all $0 \leq y_1 < y_2 < \infty$. Thus there are numbers $0 \leq y_1 < y_2 < y_3 < y_4$ such that $G(y_1), G(y_3) \notin H$ and $G(y_2), G(y_4) \in H$. Now let $N$ be large enough so that...
Let $F^{-N}(G(y_i)) \in W^u_{loc}(p)$ for $i = 1, 2, 3, 4$ and let $0 \leq x^1_m < x^2_m < x^3_m < x^4_m \leq 1$ be such that $\psi((0, \ldots, x^i_m)) = F^{-N}(G(y_i))$ for $i = 1, 2, 3, 4$. Since $F^{-N}(H)$ is closed in $M$ and $F^{-N}(G(y_1)), F^{-N}(G(y_3)) \notin F^{-N}(H)$, there is an $r$, $0 < r \leq 1$, such that

$$
\psi \left( \left\{ \left( x_1, \ldots, x_{m-1}, x^i_m \right) \mid \sum_{j=1}^{m-1} x^2_j \leq r \right\} \right) \cap F^{-N}(H) = \emptyset
$$

for $i = 1$ and $i = 3$.

Given $r$ as above, let $n = n(r)$ be as in the first paragraph of this proof. Let $D = \{(x_1, \ldots, x_m)\mid (x_1, \ldots, x_m) \in C_n$ and $x^1_m \leq x^2_m \leq x^3_m$, or $\sum_{j=1}^{m-1} x^2_j \leq r$ and $x^1_m = x^2_m$, or $\sum_{j=1}^{m-1} x^2_j \leq r$ and $x^3_m = x^4_m \}$. Then $\psi(D) \cap F^{-N}(H) = \emptyset$ and $\psi(D)$ separates $f^{-N}(G(y_2))$ from $f^{-N}(G(y_4))$ in $M$. But then $F^{-N}(H)$, and hence $H$ itself, is not connected.

Thus, no proper subcontinuum of $cl(W^u+(p))$ has nonempty interior and $cl(W^u+(p))$ is indecomposable.

**III.** In this section we apply our theorem to obtain the corollary mentioned in the Introduction.

Given a continuum $X$ and a point $x \in X$, the composant determined by $x, C_x$, is the union of all proper subcontinua of $X$ that contain $x$. If $X$ is indecomposable, the composants of $X$ partition $X$, there are uncountably many distinct composants, and each is dense in $X$ (see [HY]).

By a topological branched one-manifold we will mean a compact, connected metric space that is locally homeomorphic to a one-point union of finitely many arcs (open, closed, or half-open, half-closed). Given a branched one-manifold $K$ with metric $d$ and a continuous map $f : K \to K$, the inverse limit space $(K, f)$ is the space $(K, f) = \{(x_0, x_1, \ldots)\mid x_n \in K$ and $f(x_{n+1}) = x_n$ for all $n \geq 0\}$ with metric $d$ given by

$$
d(x, y) = \sum_{n=0}^{\infty} \frac{d(x_n, y_n)}{2^n}.
$$

$(K, f)$ is a continuum and the induced map on $(K, f)$ is the homeomorphism (onto) $\hat{f} : (K, f) \to (K, f)$ given by $\hat{f}((x_0, x_1, \ldots)) = (f(x_0), x_0, \ldots)$.

Maps $f : X \to X$ and $g : Y \to Y$ of the topological spaces $X$ and $Y$ are said to be topologically conjugate if there is a homeomorphism $h$ from $X$ onto $Y$ such that $h^{-1} \circ g \circ h = f$.

In the corollary, we have in mind the dynamical situation pictured in Figure 1.

**COROLLARY.** Let $F$ be as in the theorem and in addition assume that $cl(W^u+(p)) \cap W^s(p) \subseteq W^u+(p)$. Then for no continuous map $f$ of a branched one-manifold $K$ is $F|_{cl(W^u+(p))}$ topologically conjugate to the induced map $\hat{f}$.

**PROOF.** Suppose there is a branched one-manifold $K$, a continuous map $f : K \to K$, and a homeomorphism $h$ from $cl(W^u+(p))$ onto $(K, f)$ such that $h^{-1} \circ f \circ h = F|_{cl(W^u+(p))}$. Without loss of generality, we may assume that $f$ is onto, otherwise replace $K$ by $L = \bigcap_{n \geq 0} f^n(L)$ $(\hat{f} : (K, f) \to (K, f)$ and $\hat{f}|_L = (L, f|_L)$ are topologically conjugate).
Since $W^u+(p) = \bigcup_{n \geq 0} G([0, n])$ we see that the composant of $cl(W^u+(p))$ determined by $p$ in $cl(W^u+(p)), C_p,$ contains $W^u+(p)$. Thus, the additional assumption in this corollary means that the stable set of $p$ in $cl(W^u+(p))$, that is \( \{x \in W^u+(p)|F^n(x) \to p \text{ as } n \to \infty \} \), is contained in $C_p$.

Thus, $S(h(p)) = \{x \in (K, f)|f(x) \to h(p) \text{ as } n \to \infty \}$ must be contained in $C_{h(p)}$, the composant determined by $h(p)$ in $(K, f)$.

Since $h$ conjugates $\hat{f}$ and $F|_{cl(W^u+(p))}$ and $p$ is a fixed point of $G$, $h(p)$ must be fixed by $\hat{f}$ so that $h(p) = (p_0, p_0, \ldots)$ for some $p_0 \in K$.

It is clear from the definition of a branched one-manifold that there is an $l < \infty$ (depending of $K$) such that any collection of $l$ distinct points in $K$ separates $K$. Now let $x = (x_0, x_1, \ldots) \in S(h(p)) \setminus \{h(p)\}$. Then $x, \hat{f}(x), \ldots, \hat{f}^{l-1}(x)$ are $l$ distinct points in $S(h(p))$. It follows that there is an $N \geq 0$ such that $\pi_N(\hat{f}^i(x)) \neq \pi(\hat{f}^j(x))$ for $0 \leq i < j \leq l - 1$ where $\pi_N$ is projection onto the $N$th coordinate. Thus $K - \{\pi_N(x), \ldots, \pi_N(\hat{f}^{l-1}(x))\} = K - \{x_N, f(x_N), \ldots, f^{l-1}(x_N)\}$ is not connected.

By the theorem, $(K, f)$ is indecomposable. Thus there is a composant $C$ in $(K, f)$ such that $C \cap C_{h(p)} = \emptyset$. In addition, $C$ is dense in $(K, f)$ and is connected. $\pi_N$ is continuous so that $\pi_N(C)$ is connected and dense in $K$. It must then be the case that $\pi_N(C) \cap \{x_n, \ldots, f^{l-1}(x_N)\} \neq \emptyset$. Say $y = (y_0, y_1, \ldots) \in C$ and $i, 0 \leq i \leq l - 1$, are such that $\pi_N(y) - y_N = f^i(x_N)$.

Now, since $x \in S(h(p))$, $\lim_{n \to \infty} f^n(x_0) = p_0$. Thus,

$$
\lim_{n \to \infty} f^n(y_0) = \lim_{n \to \infty} f^{n+N}(f^i(x_N)) = \lim_{n \to \infty} f^{n+i}(x_0) = p_0
$$

and it follows that $y \in S(h(p))$. But this is impossible since $y \in C$, $C \cap C_{h(p)} = \emptyset$ and $S(h(p)) \subseteq C_{h(p)}$. Thus, there can be no such $K, f$, and $h$.

REFERENCES


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