ON 3-MANIFOLDS HAVING SURFACE-BUNDLES AS BRANCHED COVERINGS

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(Communicated by Haynes R. Miller)

To Professor F. Botella on his seventieth birthday

ABSTRACT. We give a different proof of the result of Sakuma that every closed, oriented 3-manifold $M$ has a 2-fold branched covering space $N$ which is a surface bundle over $S^1$. We also give a new proof of the result of Brooks that $N$ can be made hyperbolic. We give examples of irreducible 3-manifolds which can be represented as $2m$-fold cyclic branched coverings of $S^3$ for a number of different $m$'s as big as we like.

1. In [S] Sakuma proves that for every closed, oriented, connected 3-manifold $M^3$, there exists an $F_g$-bundle over $S^1$, $W^3$, where $F_g$ is a closed, oriented and connected surface of genus $g$, such that $W^3$ is a 2-fold branched covering of $M^3$.

He shows this by thinking of a handlebody $X_g$, of genus $g$, as the mapping cylinder of $f: F_g \rightarrow P_g$ (defined in Figure 1).

If $M$ has a Heegaard splitting $M = X_g \cup X'_g$, there is a 2-fold covering of $M$ branched over $\partial P_g \cup \partial P'_g$ that can be constructed by splitting $M$ along $P_g \cup P'_g$ and pasting together two copies of the resulting $F_g$-bundle over $[0,1]$. This 2-fold covering is $W^3$.

2. In this note we give an alternative proof of the same result. Namely, we show

**Lemma 1.** Let $M^3$ be a closed, oriented 3-manifold having an open book structure, $M^3 = M(F_{g,h}; \phi)$, where $F_{g,h}$ is a compact, connected, and oriented surface of genus $g$ with $h$ boundary components, and where $\phi: F_{g,h} \rightarrow F_{g,h}$ (the monodromy map) is a homeomorphism which restricts to the identity map on the boundary of $F_{g,h}$. Then there exists an $F_k$-bundle over $S^1$, $M(\# \phi^{-1})$, which is a 2-fold covering of $M(F_{g,h}; \phi)$ branched over a $2h$-component link, and where $k = 2g + h - 1$.

Since every $M^3$ is an open book $M(F_{g,1}; \phi)$ [GA, M] we deduce

**Corollary 2.** Every closed, oriented connected 3-manifold $M^3$ contains a 2-component link $L$, such that there is a 2-fold covering of $M^3$ branched over $L$ which is an $F_k$-bundle over $S^1$.

**Proof of Lemma 1.** Let $2F_{g,h}$ be the double of $F_{g,h}$ and let $i: 2F_{g,h} \rightarrow 2F_{g,h}$ be the natural involution interchanging the two copies of $2F_{g,h}$.
Let $M(\phi \# \phi^{-1})$ be the $2F_{g,h}$-bundle over $S^1$ with monodromy $\phi \# \phi^{-1}$ defined by

$$
\frac{2F_{g,h} \times [-1,1]}{(x,1) \equiv (\phi \# \phi^{-1} x, -1)},
$$

where $\phi \# \phi^{-1}: 2F_{g,h} \to 2F_{g,h}$ is $\phi$ in one copy of $F_{g,h}$, and is $\phi^{-1}$ in the other copy.

Consider the involution $u: M(\phi \# \phi^{-1}) \to M(\phi \# \phi^{-1})$ given by $u(x,t) = (ix, -t)$ for every $(x, t) \in 2F_{g,h} \times [-1,1]$. It has a pair of double curves for each component of $\partial F_{g,h}$, namely $\text{Fix } u = \partial F_{g,h} \times \{0,1\}$.

The quotient of $M(\phi \# \phi^{-1})$ under $u$ is the quotient of $F_{g,h} \times [-1,1]/((x, 1) \equiv (x, -1))$ under the identification $(x, t) \equiv (x, -t)$ for every $(x, t) \in (\partial F_{g,h} \times [-1,1])/((x, 1) \equiv (x, -1))$.

This is equivalent to identifying $x \times [0,1]$ with $x \times [0, -1]$, for every $x \in \partial F_{g,h}$, or to collapsing $x \times S^1$ into a point for every $x \in \partial F_{g,h}$. Therefore, the quotient of $M(\phi \# \phi^{-1})$ under the action of $u$ is the open book $M(F_{g,h}; \phi)$. □

REMARK. The branching set of the 2-fold covering of the lemma is the boundary of a collar of a leaf.

EXAMPLE. The figure-eight knot 4\_1 in $S^3$ is a fibered knot. The 2-fold covering of $S^3$ branched over the union of 4\_1 and its canonical longitude (i.e. the boundary of a collar of a fiber) is an $F_2$-bundle over $S^1$ with monodromy $\phi \# \phi^{-1}$, where $\phi = [2 \ 1 \ 1 \ 1]$ on a torus with a hole. This 2-fold covering is the result of 0-Dehn surgery on 4\_1 # 4\_1, and this representation helps to visualize the $F_2$-bundle structure. As a consequence of this example, note that the $m$-cyclic covering of $S^3$ branched over 4\_1 is the quotient of $M(\phi^m \# \phi^{-m})$ under the action of $u$ (defined in the proof of the lemma).

Now consider the link $L_r(F_{g,h}; \phi)$ in $M(F_{g,h}; \phi)$ formed by the boundary of a collar $C$ of a leaf of $M(F_{g,h}; \phi)$ together with the union of $r$ sections of $M(F_{g,h}; \phi)$ not intersecting $C$ (see $L_3(F_{1,1}; [2 \ 1 \ 1 \ 1])$ in Figure 2). For each $m > 1$ there is a regular covering of $M(F_{g,h}; \phi)$, branched over $L_r(F_{g,h}; \phi)$, with group of covering translations $\mathbb{Z}_2 \times \mathbb{Z}_m$ which is the composition of the 2-fold covering $p: M(\phi \# \phi^{-1}) \to M(F_{g,h}; \phi)$.
defined in the proof of the lemma, with an \( m \)-cyclic covering

\[
q : W \rightarrow M(\phi \# \phi^{-1})
\]

branched over \( p^{-1}(L_r(F_{g,h}; \phi) \setminus \partial C) \). The manifold \( W \) is an \( F_k \)-bundle over \( S^1 \), and the genus \( k \) of the fiber \( F_k \) increases as \( m \) or \( r \) increases. Thus

**COROLLARY 3.** Every closed, oriented 3-manifold admits \( 2m \)-fold cyclic branched coverings, \( m = \text{odd} \), which are \( F_k \)-bundles over \( S^1 \), and where \( m \) and \( k \) are as big as we like. \( \square \)

**REMARK.** This observation could have been easily obtained using the method of Sakuma [S], by considering Heegaard splittings of arbitrarily big genus. But with our method we gain control on the branching set, as we see in the next example.

**EXAMPLE.** The \( 2m \)-fold branched covering corresponding to \( L_r(F_{0,0}; \text{id}) \) is \( S^1 \times F_h \), where \( h = (m - 1)(r - 1) \), \( m > 1, \ r \geq 0 \) (Figure 3). This is a cyclic covering if and only if \( m = \text{odd} \). Thus \( S^1 \times F_2 \) is a 6-fold cyclic covering of \( S^3 \) [HN]. More generally:

**COROLLARY 4.** \( S^1 \times F_{2^y} \) is a cyclic branched covering of \( S^3 \) of \( 2(2^y + 1) \) sheets, for every \( y \leq x \).

Thus we see that there are irreducible 3-manifolds which can be represented as \( 2m \)-fold cyclic branched coverings of \( S^3 \) for a number of different \( m \)'s as big as we like.

We end this paper with three questions.

**QUESTION 1.** Are there closed, orientable, irreducible 3-manifolds which are \( m \)-fold cyclic branched coverings of \( S^3 \) for as many primes \( m \) as we like?

Since the branching sets \( L_r(F_{0,0}; \text{id}) \) have bridge number \( r + 2 \), one would expect that the answer to the next question would be in the affirmative.

**QUESTION 2.** Does \( S^1 \times F_{2^x} \) have at least \( x \) different minimal Heegaard splittings (i.e. no trivial handles) all of them of different genus? (Compare [CG].)

Corollary 3 suggests the next question.
QUESTION 3. Is there a prime \( p \neq 2 \) such that every closed, oriented 3-manifold has a branched \( p \)-fold cyclic covering which is an \( F_p \)-bundle over \( S^1 \)?

ADDED IN JULY 1986. After this paper was accepted I have seen [B] where, using Sakuma’s method, it is shown that every closed, oriented connected 3-manifold has a 2-fold branched covering which is a hyperbolic manifold and an \( F_2 \)-bundle over \( S^1 \). The same result can be obtained using the methods of this paper. To see this, we follow the notation of Lemma 1. First take \( F_{g,1} \) to be the fiber of a hyperbolic fibered knot \( K = \partial F_{g,1} \) in \( M^3 \) [So]. Then the boundary \( K \cup K' \) of a collar \( A \) of \( F_{g,1} \) is the branching set of a 2-fold covering \( M(\phi \# \phi^{-1}) \) where \( \phi \) is the monodromy of \( K \). Since \( \phi \) is pseudo-Anosov there exists a simple arc \( \gamma \) properly embedded in \( F \setminus \text{Int} A \) (i.e. \( \partial \gamma \subset K' \)) such that the orbit of \( \gamma \) under \( \phi \) fills \( F \setminus \text{Int} A \) [F]. Modifying \( K' \) suitably in a regular neighborhood of \( \gamma \) changes the 2-fold covering \( M(\phi \# \phi^{-1}) \) by \( \frac{1}{n} \)-Dehn surgery on \( \gamma \) covering \( \hat{\gamma} \) [Mo]. Here \( \gamma \) is a simple closed curve which doubles \( \hat{\gamma} \) in \( 2F_{g,1} \), the fiber of \( M(\phi \# \phi^{-1}) \). The manifold resulting from this \( \frac{1}{n} \)-Dehn surgery on \( M(\phi \# \phi^{-1}) \) still is a 2-fold covering of \( M \), branched over \( K \cup \hat{K} \) (\( K \) modified along \( \hat{\gamma} \)), and a \( 2F_{g,1} \)-bundle over \( S^1 \), but the monodromy is the composition of \( \phi \# \phi^{-1} \) with \( T^n \), the \( n \)-th power of a Dehn twist along \( \gamma \) [St].

Since the orbit of \( \gamma \) under \( \phi \# \phi^{-1} \) fills \( 2F_{g,1} \), it follows from [F] that \( T^n(\phi \# \phi^{-1}) \) is pseudo-Anosov except for at most seven consecutive values of \( n \). This finishes the proof of the theorem.

The last proof was obtained with the generous help of F. Bonahon and M. Boileau. It is a pleasure to record here my warmest thanks to both of them.

REFERENCES


[GA] F. González-Acuña, 3-dimensional open books, Lectures, Univ. of Iowa, Topological Seminar, 1974/75.


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