THE PREPARATION THEOREM
FOR COMPOSITE FUNCTIONS

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ABSTRACT. We present a simple extension of the preparation theorem of B. Malgrange and J. Mather to the case of composite functions. As a corollary we obtain a short proof of the equivariant preparation theorem of V. Poénaru.

1. Formulation of the results. We denote by \( \mathcal{E}(p) \) the ring of germs of \( C^\infty \)-functions at \( 0 \in \mathbb{R}^p \), by \( \mathcal{E}(p,q) \) the set of germs at \( 0 \in \mathbb{R}^p \) of \( C^\infty \)-transformations \( \mathbb{R}^p \to \mathbb{R}^q \) which preserve the origin, and by \( m(p) \) the maximal ideal of \( \mathcal{E}(p) \) formed by all functions vanishing at 0. A transformation \( H \in \mathcal{E}(p,q) \) induces a local ring homomorphism \( \mathcal{E}(q) \to \mathcal{E}(p) \) defined by \( \beta \mapsto \beta \circ H \).

For \( p \in \mathcal{E}(m,k) \) and \( \eta \in \mathcal{E}(n,l) \), we introduce \( \mathcal{E}_\rho(m) \overset{\text{def}}{=} \{ \alpha \circ \rho; \alpha \in \mathcal{E}(k) \} \), \( \mathcal{E}_\eta(n) \overset{\text{def}}{=} \{ \beta \circ \eta; \beta \in \mathcal{E}(l) \} \), \( m_\rho(m) \overset{\text{def}}{=} m(m) \cap \mathcal{E}_\rho(m) \), and \( m_\eta(n) \overset{\text{def}}{=} m(n) \cap \mathcal{E}_\eta(n) \). Obviously \( m_\eta(n) = \eta^* m(l) \) and \( m_\rho(m) = \rho^* m(k) \).

A germ \( f \in \mathcal{E}(m,n) \) such that \( f^* \mathcal{E}_\eta(n) \subset \mathcal{E}_\rho(m) \) will be called a \( \rho \eta \)-germ; for such a transformation each component of \( \eta \circ f \) belongs to \( \mathcal{E}_\rho(m) \) and so is of the form \( F \circ \rho \). Hence there exists \( F \in \mathcal{E}(k,l) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R}^k & \overset{F}{\to} & \mathbb{R}^l \\
\uparrow \rho & & \uparrow \eta \\
\mathbb{R}^m & \overset{f}{\to} & \mathbb{R}^n
\end{array}
\]

As \( \rho^* \mathcal{E}(k) \) and \( f^* \mathcal{E}_\eta(n) = (\eta \circ f)^* \mathcal{E}(l) = (F \circ \rho)^* \mathcal{E}(l) \) are all subrings of \( \mathcal{E}_\rho(m) \), any \( \mathcal{E}_\rho(m) \)-module \( A \) could be considered as an \( \mathcal{E}(k) \)-, \( \mathcal{E}_\rho(m) \)- or \( \mathcal{E}(l) \)-module. Obviously \( a_1, \ldots, a_p \) generate \( A \) as an \( \mathcal{E}_\rho(m) \)-module (respectively an \( \mathcal{E}_\eta(n) \)-module) if and only if they generate it as an \( \mathcal{E}(k) \)-module (\( \mathcal{E}(l) \)-module, respectively). A similar remark concerns the generators of the isomorphic vector spaces

\[
A/F^* m(l) \cdot A \approx A/(\eta \circ f)^* m(l) \cdot A \approx A/f^* m_\eta(n) \cdot A.
\]

From the preparation theorem [1, p. 59, 2, 3] applied to \( F \) we obtain the following result.

THEOREM 1. Let \( \rho \in \mathcal{E}(m,k) \), \( \eta \in \mathcal{E}(n,l) \), let \( A \) be a finitely generated \( \mathcal{E}_\rho(m) \)-module, let \( f \in \mathcal{E}(m,n) \) be a \( \rho \eta \)-germ, and suppose \( F \in \mathcal{E}(k,l) \) makes the diagram commutative.

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(1) commute. Then the elements $a_1, \ldots, a_p$ generate $A$ as an $\mathcal{E}_n(n)$-module if and only if they represent generators of the real vector space $A/(f^*m_n(n) \cdot A)$.

Let $m(\rho, k)$ denote the ideal in $\mathcal{E}(k)$ of all functions vanishing on $\rho(\mathbb{R}^m)$. Note that the ring $\mathcal{E}_\rho(m)$ is isomorphic to $\mathcal{E}(k)/m(\rho, k)$. Hence for any $\mathcal{E}(k)$-module $A^*$ the factor module $A^*/(m(\rho, k) \cdot A^*)$ has the natural structure of an $\mathcal{E}_\rho(m)$-module.

**Corollary 1.** With the hypotheses of Theorem 1 let $A \overset{\text{def}}{=} A^*/m(\rho, k) \cdot A^*$, where $A^*$ is a finitely generated $\mathcal{E}(k)$-module. Then $a_1, \ldots, a_p$ belonging to $A^*$ represent generators of the $\mathcal{E}_n(n)$-module $A$ if and only if they represent generators of the real vector space $A^*/(m(\rho, k) \cdot A^* + F^*m(l) \cdot A^*)$.

**Proof.** Let us denote by $A_1$ the $\mathcal{E}(k)$-submodule $m(\rho, k) \cdot A^*$ of $A^*$. From the following sequence of the natural $\mathcal{E}(k)$-module isomorphisms
\[
A/f^*m_n(n) \cdot A = A/\rho^*F^*m(l) \cdot A \cong A/F^*m(l) \cdot A
\]
\[= (A^*/A_1)/((F^*m(l) \cdot A^*)/A_1) \cong A^*/(F^*m(l) \cdot A^* + A_1),
\]

it follows that the vector spaces $A/f^*m_n(n) \cdot A$ and $A^*/(F^*m(l) \cdot A^* + m(\rho, k) \cdot A^*)$ are isomorphic. Now we can refer to Theorem 1, since $A$ is a finitely generated $\mathcal{E}_\rho(m)$-module (because $A^*$ is finitely generated over $\mathcal{E}(k)$).

**2. Equivariant division theorem.** This paragraph provides some examples of applications of Theorem 1.

Consider a compact Lie group $G$ acting orthogonally on $\mathbb{R}^m$ and $\mathbb{R}^n$. According to G. Schwarz [5] there exist polynomial maps $\rho \in \mathcal{E}(m, k)$ and $\eta \in \mathcal{E}(n, l)$, called Hilbert maps, such that $\mathcal{E}_\rho(m)$ and $\mathcal{E}_\eta(n)$ are exactly the sets of $G$-invariant germs $\mathcal{E}_G(m)$ and $\mathcal{E}_G(n)$, respectively. Denote $m_G(n) \overset{\text{def}}{=} m(n) \cap \mathcal{E}_G(n)$. Obviously any $G$-equivariant $f \in \mathcal{E}(m, n)$ is a $\rho\eta$-germ, so from Theorem 1 there follows the equivariant preparation theorem [4].

**Theorem 2.** If $A$ is a finitely generated $\mathcal{E}_\rho(m)$-module then $A$ is a finitely generated as a $\mathcal{E}_G(n)$-module if and only if the real vector space $A/f^*m_G(n)A$ has a finite dimension.

**Example.** Let $\mathbb{R}^m = \mathbb{R}^n = \mathbb{R}^2$. Let $G = Z_2 = \{\pm 1\}$ operate on $\mathbb{R}^2$ as $(x, y) \mapsto (\varepsilon x, \varepsilon y)$ for $\varepsilon \in G$. Let us consider an equivariant transformation $f: \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x, x^3 + y^3)$. Using Corollary 1 we shall show that $1, xy, y^2, y^4$ generate $\mathcal{E}_{Z_2}(2)$ over $f^*\mathcal{E}_{Z_2}(2)$.

The Hilbert maps $\rho = \eta: \mathbb{R}^2 \to \mathbb{R}^3$ could be defined as $(x, y) \mapsto (x^2, y^2, xy)$. The set $\rho(\mathbb{R}^2) \subset \mathbb{R}^3$ is a semicone $uv = z^2, u \geq 0, v \geq 0$, where $u, v, z$ are coordinates in $\mathbb{R}^3$. Obviously $\mathcal{E}_{Z_2}(2) = \rho^*\mathcal{E}(3) \cong \mathcal{E}(3)/m(\rho, 3)$. Transformation $f$ is a $\rho\rho$-germ and $\rho \circ f(x, y) = (x^2, (x^3 + y^3)^2, x^4 + xy^3)$ is a $Z_2$-invariant mapping. A mapping $F: \mathbb{R}^3 \to \mathbb{R}^3$ as in (1), i.e. such that $\rho \circ f = F \circ \rho$, could be defined by $(u, v, z) \mapsto (u, u^3 + 2z^3 + v^3, u^2 + zv)$. Let us consider the ideal $I \overset{\text{def}}{=} \langle u, z^2, zv, v^3 \rangle_{\mathcal{E}(3)}$. By straightforward checking we get
\[
I = \langle u, u^3 + 2z^3 + v^3, u^2 + zv, uv - z^2 \rangle_{\mathcal{E}(3)} \subset (F^*m(3) \cdot \mathcal{E}(3) + m(\rho, 3) \cdot \mathcal{E}(3)).
\]

It is easy to observe that $m^3(3) \subset I + m^4(3)$, so $m^3(3) \subset I$ by Nakayama’s lemma [1]. Now a simple computation shows that $1, z, v, v^2$ represent generators of
the real vector space $\mathcal{E}(3)/I$ and so they generate $\mathcal{E}(3)/(F^*m(3) \cdot \mathcal{E}(3) + m(\rho, 3))$, the real vector space. By Corollary 1 (for $A^* = \mathcal{E}(3)$ and $A = \mathcal{E}(3)/m(\rho, 3)$) they represent generators of module $A$ over $f^*\mathcal{E}_Z(2) \simeq \mathcal{E}_Z(2)$. Now considering an $f^*\mathcal{E}_Z(2)$-module isomorphism $\rho^*: A \rightarrow \mathcal{E}_Z(2)$ we find that their combinations with $\rho$, i.e. $1, xy, y^2, y^4$ generate $\mathcal{E}_Z(2)$ over $f^*\mathcal{E}_Z(2)$.

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