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ON PRIMES p WITH $\sigma(p^\alpha) = m^2$

J. CHIDAMBARASWAMY AND P. V. KRISHNAIAH

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ABSTRACT. A. Takaku proved that for odd $\alpha \geqslant 3$, $\sigma(p^\alpha) = m^2$, p being a prime, implies that $p < 2^{2^{\alpha+1}}$. In this paper we extend this result to include almost all even integers α.

Introduction. For an integer $n > 1$, let $\sigma(n)$ be the sum of all positive divisors of n. Recently, Takaku [1] proved that for any given odd integer $\alpha > 3$, all primes p such that $\sigma(p^\alpha)$ is a perfect square satisfy $p < 2^{2^{\alpha+1}}$. The purpose of this paper is to extend this result by a modification of his method to include almost all even integers α.

We write $\eta(k)$ for 2^{k-1} and $\beta(k)$ for $2^k - 2$. We define the "sequences" $\{\delta_i^{(k)}: k \geqslant 2, 2 \leqslant i \leqslant k\}$ and $\{\gamma_i^{(k)}: k \geqslant 2, 2 \leqslant i \leqslant k\}$ by means of the following system of simultaneous recurrence relations and initial conditions:

$$\delta_2^{(2)} = 1, \qquad \gamma_2^{(2)} = -2, \tag{1.1}$$

$$\begin{aligned} &\text{(a)} \quad \delta_2^{(k+1)} = \left(2^{\beta(k)} - \delta_k^{(k)}\right)^2, \\ &\text{(b)} \quad \delta_3^{(k+1)} = -2^{\eta(k)}\gamma_2^{(k)}\left(2^{\beta(k)} - \delta_k^{(k)}\right), \\ &\text{(c)} \quad \delta_{i+1}^{(k+1)} = -2^{\eta(k)}\gamma_i^{(k)}\left(2^{\beta(k)} - \delta_k^{(k)}\right) + 2^{2\eta(k)}\delta_{i-1}^{(k)}, \qquad 3 \leqslant i \leqslant k, \end{aligned} \tag{1.2}$$

$$\begin{aligned} &\text{(a)} \quad \gamma_2^{(k+1)} = -2\left(2^{\beta(k)} - \delta_k^{(k)}\right), \\ &\text{(b)} \quad \gamma_i^{(k+1)} = 2^{\eta(k)}\gamma_{i-1}^{(k)}, \qquad 3 \leqslant i \leqslant k+1. \end{aligned} \tag{1.3}$$

THEOREM. *Let* $A = \{k \mid k > 2, \delta_2^{(k)} = \cdots = \delta_{k-1}^{(k)} = 0, \delta_k^{(k)} = 2^{\beta(k)}\}$. *Then*

(1) *no odd integer belongs to* A,

(2) *the density of* A *in the set of all positive integers is zero, and*

(3) *for any given integer* α *not belonging to* A, *all primes* p *for which* $\sigma(p^\alpha)$ *is a perfect square satisfy* $p < 2^{2^{\alpha+1}}$.

REMARK. Takaku obtained the conclusion (3) above for odd integers $\alpha > 3$.

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The second author is on leave from Andhra University, Waltair, India.

2. Proof of the Theorem. We need the following lemmas. In Lemma 1 we assume that p is a prime, $\alpha \geqslant 3$, and $\sigma(p^\alpha) = m^2$.

LEMMA 1. For $2 \leqslant k \leqslant \alpha$, there exists an integer x_k such that

(2.1)
$$2^{\beta(k)}(1 + p + \cdots + p^{\alpha-k}) = p^{k-2}(px_k)^2 + \sum_{i=2}^{k} p^{k-i}\{\gamma_i^{(k)}px_k + \delta_i^{(k)}\} - 2^{\eta(k)}x_k.$$

PROOF. We use induction on k. By our assumption on p and α, we have $p + p^2 + \cdots + p^\alpha = (m+1)(m-1)$, so that $p \mid m + \varepsilon$, where $\varepsilon = \pm 1$. We write $m + \varepsilon = px_1$, $x_1 > 0$, and note that $(m+1)(m-1) = px_1(px_1 - 2\varepsilon)$, so that we have $p + p^2 + \cdots + p^{\alpha-1} = px_1^2 - 2\varepsilon x_1 - 1$. Now writing $px_2 = 2\varepsilon x_1 + 1$ (the fact that $p \mid 2\varepsilon x_1 + 1$ shows that p must be an odd prime), we have

$$(2.2)\quad 2^2(1 + p + p^2 + \cdots + p^{\alpha-2}) = 2^2x_1^2 - 2^2x_2 = (px_2 - 1)^2 - 2^2x_2$$
$$= (px_2)^2 + \gamma_2^{(2)}px_2 + \delta_2^{(2)} - 2^2x_2$$

which is (2.1) for $k = 2$. We refer to Lemma 2 of **[1]** for the details of induction.

REMARK. In Lemma 1 of **[1]** px_2 is defined as $2x_1 + \varepsilon$ ($\varepsilon = 1$ or -1 according as $p \mid m + 1$ or $p \mid m - 1$) and hence ε is present in the statement of that lemma. Consequently, the numbers $\gamma_i^{(k)}$ and $\delta_i^{(k)}$ of Lemma 2 of **[1]** might appear to depend on ε and hence on α, p, and m. However, if px_2 is defined as $2\varepsilon x_1 + 1$ as we have done in Lemma 1 above, the ε will disappear and the numbers $\gamma_i^{(k)}$ and $\delta_i^{(k)}$ will indeed be independent of α, p, and m as mentioned by Takaku in Lemma 2 of **[1]**.

LEMMA 2. *For $2 \leqslant k \leqslant \alpha$, $2 \leqslant i \leqslant k$, we have $|\gamma_i^{(k)}| < 2^{1+2^k}$ and $|\delta_i^{(k)}| < 2^{2^{k+1}-1}$.*

For $k \geqslant 3$, this is Lemma 4 of **[1]**. For $k = 2$ this is clear in virtue of (1.1).

LEMMA 3. *If $l > 3$ and*
$$\delta_2^{(l)} = \delta_3^{(l)} = \cdots = \delta_j^{(l)} = 0, \qquad \delta_{j+1}^{(l)} = \cdots = \delta_l^{(l)} = 2^{\beta(l)}$$
for $2 < j < l$, then
$$\delta_2^{(l-1)} = \cdots = \delta_{j-2}^{(l-1)} = 0 \quad and \quad \delta_{j-1}^{(l-1)} = \cdots = \delta_{l-1}^{(l-1)} = 2^{\beta(l-1)}.$$

PROOF. We have, by (1.2)(a),
$$\left(2^{\beta(l-1)} - \delta_{l-1}^{(l-1)}\right)^2 = \delta_2^{(l)} = 0$$
so that by (1.2)(c),
$$\delta_{i+1}^{(l)} = 2^{2\eta(l-1)}\delta_{i-1}^{(l-1)} \quad \text{for } 3 \leqslant i \leqslant l-1.$$
This, in view of our hypothesis, implies that $\delta_2^{(l-1)} = \cdots = \delta_{j-2}^{(l-1)} = 0$ and for $j \leqslant i \leqslant l-1$
$$\delta_{i-1}^{(l-1)} = 2^{-2\eta(l-1)}\delta_{i+1}^{(l)} = 2^{\beta(l)-2\eta(l-1)} = 2^{\beta(l-1)} = \delta_{l-1}^{(l-1)}.$$

LEMMA 4. *If $k \in A$, k must be even and for $r = k/2 + 1$,*
(a) $\delta_2^{(r)} = \delta_3^{(r)} = \cdots = \delta_r^{(r)} = 2^{\beta(r)}$, *and*
(b) $n \notin A$ *for* $r \leqslant n \leqslant k-1$.

PROOF. Let $k \in A$ and, if possible, let $k = 2l + 1$ so that $l \geqslant 1$. Applying Lemma 3 successively $l - 1$ times, we get

$$\delta_2^{(l+2)} = 0,\ \delta_3^{l+2} = \cdots = \delta_{l+2}^{l+2} = 2^{\beta(l+2)} \neq 0.$$

But, by (a) of (1.2) $\delta_2^{(l+2)} = 0$ implies $\delta_{l+1}^{(l+1)} = 2^{\beta(l+1)}$ and this in turn implies by (b) of (1.2) that $\delta_3^{(l+2)} = 0$, which is a contradiction. Hence k is even and $\geqslant 4$.

Now, applying Lemma 3 successively $r - 2$ times we get (a). The truth of (b) follows by an observation of $\delta_2^{(n)}, \delta_3^{(n)}, \ldots, \delta_n^{(n)}$ for $r \leqslant n \leqslant k - 1$; in fact, if $k \in A$ applying Lemma 3 u times $1 \leqslant u \leqslant k/2 - 1$, we get $\delta_{k-u-1}^{(k-u)} \neq 0$ and so $k - u \notin A$.

LEMMA 5. *If* $k \notin A, \sum_{i=2}^k p^{k-i}\delta_i^{(k)} - 2^{\beta(k)} \neq 0$ *for* $p > 2^{2^{k+1}}$.

PROOF. If $\delta_2^{(k)} = \delta_3^{(k)} = \cdots = \delta_{k-1}^{(k)} = 0$, then, since $k \notin A$, $\delta_k^{(k)} \neq 2^{\beta(k)}$ and the result follows.

If $\delta_2^{(k)} = \cdots = \delta_{k-2}^{(k)} = 0$ and $\delta_{k-1}^{(k)} \neq 0$, then by Lemma 2

$$\left|\delta_{k-1}^{(k)}p + \delta_k^{(k)} - 2^{\beta(k)}\right| \geqslant p - \left|\delta_k^{(k)}\right| - 2^{\beta(k)} > 2^{2^{k+1}} - 2^{2^{k+1}-1} - 2^{2^k-2} > 0.$$

Suppose then $\delta_j^{(k)} \neq 0$ for some j, $2 \leqslant j \leqslant k - 2$, and $\delta_i^{(k)} = 0$ for $i < j$. Then

$$\begin{aligned}\left|\sum_{i=j}^{k} p^{k-i}\delta_i^{(k)} - 2^{\beta(k)}\right| &\geqslant p^{k-j}\left|\delta_j^{(k)}\right| - \sum_{i=j+1}^{k} p^{k-i}\left|\delta_i^{(k)}\right| - 2^{\beta(k)} \\ &\geqslant p^{k-j-2}\left\{p^2 - p\left|\delta_{j+1}^{(k)}\right| - \sum_{i=j+2}^{k}\left|\delta_i^{(k)}\right| - 2^{\beta(k)}\right\} \\ &> p^{k-j-2}\left\{p\left(2^{2^{k+1}} - 2^{2^{k+1}-1}\right) - (k-j)2^{2^{k+1}-1}\right\} \\ &> p^{k-j-2}\left\{2^{2^{k+1}} \cdot 2^{2^{k+1}-1} - (k-j)2^{2^{k+1}-1}\right\} \\ &= p^{k-j-2} \cdot 2^{2^{k+1}-1}\left\{2^{2^{k+1}} - (k-j)\right\} > 0.\end{aligned}$$

PROOF OF THE THEOREM. (1) follows from Lemma 4.

(2) We write A as an increasing sequence $\{k_n, n \geqslant 1\}$ and put $r_n = \frac{1}{2}k_n + 1$. Then for each $n \geqslant 1$, by (b) of Lemma 4, we get $k_n \leqslant r_{n+1} - 1 = \frac{1}{2}k_{n+1}$ and (2) follows.

(3) Let $\alpha \geqslant 2$, $\alpha \notin A$, $p > 2^{2^{\alpha+1}}$, and $\sigma(p^\alpha) = m^2$. Putting $k = \alpha$ in Lemma 1, we obtain for an integer x_α

$$(2.3) \qquad p^{\alpha-2}(px_\alpha)^2 + \sum_{i=2}^{\alpha} p^{\alpha-i}\left\{\gamma_i^{(\alpha)}px_\alpha + \delta_i^{(\alpha)}\right\} - 2^{\eta(\alpha)}x_\alpha - 2^{\beta(\alpha)} = 0.$$

The expression on the l.h.s. of (2.3) is a quadratic in x_α whose constant term is $\neq 0$ by Lemma 5 and hence x_α is a nonzero integer. Now we show that the leading coefficient p^α in (2.3) is greater than the sum of the absolute values of the other two coefficients and this would imply that $|x_\alpha| < 1$, which is a contradiction. Since

$\eta(\alpha) < \beta(\alpha)$, we have, using Lemma 2,

$$\begin{aligned}\left|\sum_{i=2}^{\alpha} p^{\alpha-i+1}\gamma_i^{(\alpha)} - 2^{\eta(\alpha)}\right| + \left|\sum_{i=2}^{\alpha} p^{\alpha-i}\delta_i^{(\alpha)} - 2^{\beta(\alpha)}\right| \\ \leqslant p^{\alpha-1}\left(\sum_{i=2}^{\alpha}\left|\gamma_i^{(\alpha)}\right|\right) + p^{\alpha-2}\left(\sum_{i=2}^{\alpha}\left|\delta_i^{(\alpha)}\right|\right) + 2^{2\beta(\alpha)} \\ < p^{\alpha-1}\cdot\alpha\cdot 2^{1+2^{\alpha}} + p^{\alpha-2}\cdot\alpha\cdot 2^{2^{\alpha+1}-1} \\ < p^{\alpha}\left(\frac{2\alpha\cdot 2^{2^{\alpha}}}{p} + \frac{\alpha}{p}\right) < p^{\alpha}\left(\frac{2\alpha}{2^{2^{\alpha}}} + \frac{\alpha}{2^{2^{\alpha+1}}}\right) < p^{\alpha}\end{aligned}$$

and the proof of the theorem is complete.

References

1. A. Takaku, *Prime numbers such that the sums of the divisors of their powers are perfect squares*, Colloq. Math. **49** (1984), 117–121.

Department of Mathematics, The University of Toledo, Toledo, Ohio 43606 (Current address of J. Chidambaraswamy)

Current address (P. V. Krishnaiah): Department of Mathematics, Andura University, Waltair, India