ON THE STRUCTURE OF SETS OF UNIQUENESS
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ABSTRACT. We show that every $U_0$-set is almost a $W$-set.

It may be expected that if a Borel set $E \subset \mathbb{T} \overset{\text{def}}{=} \mathbb{R}/\mathbb{Z}$ cannot carry any Borel measure $\mu$ whose Fourier-Stieltjes coefficients $\hat{\mu}(n) \overset{\text{def}}{=} \int_{\mathbb{T}} e^{-2\pi int} \, d\mu(t)$ vanish at infinity, then the arithmetic of $E$ is partially responsible. We shall show that this is precisely the case.

Recall the following definitions (see [3]).

DEFINITION. A Borel measure $\mu$ on $\mathbb{T}$ is a Rajchman measure if $\lim_{|n| \to \infty} \hat{\mu}(n) = 0$; $R$ denotes the set of Rajchman measures. A set $E$ is a set of uniqueness in the wide sense, or $U_0$-set, if $\mu_E = 0$ for all $\mu \in R$. A Borel set $E \subset \mathbb{T}$ is a $W$-set if there is some strictly increasing sequence of integers $\{n_k\}_{k=1}^\infty$ such that for all $x \in E$, $\{n_k x\}$ has a nonuniform asymptotic distribution $\nu_x$.

Let us say that a set $E$ is almost in a class $C$ if for every positive Borel measure $\mu$ carried by $E$, there is a set $F \subset C$ such that $\mu(E \setminus F) = 0$. In [3], we showed that $\mu \in R$ if and only if $\mu_E = 0$ for all $E \in W$. This immediately implies that every $U_0$-set is almost in $W_\sigma$, where $W_\sigma$ is the class of sets which are countable unions of $W$-sets. Indeed, given $E \in U_0$ and $\mu$ a positive measure carried by $E$, we have that $\sup_{G \in W_\sigma} \mu G$ is attained. Since $\mu \not\in R$ for all Borel $F \subset E$ unless $\mu F = 0$, it is easy to see that $\sup_{G \in W_\sigma} \mu G = ||\mu||$, whence the claim follows. We shall prove here the following stronger result.

THEOREM. A Borel set $E$ is a $U_0$-set if and only if $E$ is almost a $W$-set.

Of course, one direction is trivial since every $W$-set is a $U_0$-set. In the other direction, we shall prove a still stronger result. Recall [3] that $E$ is a $W_1$-set if $E$ is a $W$-set corresponding to asymptotic distributions $\nu_x$ with $\nu_x(1) \neq 0$ for $x \in E$. We shall show that $U_0$-sets are in fact almost $W_1$-sets. Furthermore, with the definitions extended as in [3], $U_0$-sets are almost $W_1$-sets in all LCA groups. For related results, see [1 and 2].

LEMMA. Let $\mu$ be a positive $\sigma$-finite measure. Suppose that $f$ and $g$ are measurable functions such that for every $x$, either $f(x) \neq 0$ or $g(x) \neq 0$. Then there exists a countable set $K \subset ]0, \infty[ \setminus K$, such that if $\alpha \in ]0, \infty[ \setminus K$, then $f(x) + \alpha g(x) \neq 0$ for $\mu$-a.e. $x$.

PROOF. Let $G_\alpha = \{x : f(x) + \alpha g(x) = 0\}$. Then $G_\alpha \cap G_\beta = \emptyset$ if $\alpha \neq \beta$, whence $K = \{\alpha > 0 : \mu G_\alpha > 0\}$ is at most countable. $\square$

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**Lemma.** Let \( \mu \) be a positive \( \sigma \)-finite measure. Suppose that \( f_n \) are measurable functions bounded by 1 such that for every \( x \), some \( f_n(x) \) is not 0. Then there exist \( \alpha_n > 0 \) such that \( \sum \alpha_n < \infty \) and \( \sum \alpha_n f_n(x) \neq 0 \) for \( \mu \)-a.e. \( x \).

**Proof.** It is easy to see that we may assume \( \mu \) to be finite. Let \( E_n = \{x: f_n(x) \neq 0\} \). We define \( \alpha_n \) inductively. Let \( \alpha_1 = 1 \). If \( \alpha_1, \ldots, \alpha_N \) have been defined, then choose \( \alpha_{N+1} < \alpha_N/2 \) such that \( \sum_{n \leq N+1} \alpha_n f_n(x) \neq 0 \) \( \mu \)-a.e. on \( \bigcup_{n \leq N+1} E_n \) and also

\[
\mu \left( \left\{ x \in \bigcup_{n \leq N} E_n : \left| \sum_{n \leq N} \alpha_n f_n(x) \right| < 2\alpha_{N+1} \right\} \right) < N^{-1}.
\]

Then if \( \sum_{n \geq 1} \alpha_n f_n(x) = 0 \), we have for all \( N \),

\[
\sum_{n \leq N} \alpha_n f_n(x) = \sum_{n > N} \alpha_n f_n(x) \leq \sum_{n > N} |\alpha_n| < 2\alpha_{N+1},
\]

whence

\[
\mu \left( \left\{ x: \sum_{n \geq 1} \alpha_n f_n(x) = 0 \right\} \right) < N^{-1} + \mu \left( \left( \bigcup_{n \leq N} E_n \right)^c \right).
\]

Since \( N \) is arbitrary, it follows that \( \sum_{n \geq 1} \alpha_n f_n(x) \neq 0 \) \( \mu \)-a.e. \( \square \)

**Remark.** It is not hard to show by using Fubini's theorem that, in fact, almost all choices of \( \{\alpha_n\} \) satisfy the lemma, where, say, \( \alpha_n \) is chosen independently and uniformly in \([0, n^{-2}]\). One may also show that except for a meager set of positive sequences in \( l^1(\mathbb{Z}^+) \), any positive sequence \( \{\alpha_n\} \) satisfies the lemma.

**Proof of the Theorem.** Let \( E \) be a \( U_0 \)-set and \( \mu \) a positive Borel measure on \( E \). Then by [3], there are \( W_1 \)-sets \( E_m \) such that \( \mu \left( E \setminus \bigcup_{m \geq 1} E_m \right) = 0 \); such that if the sequence corresponding to \( E_m \) is \( \{n_{k,m}\} \), then \( \{n_{k,m}, x\} \) has an asymptotic distribution \( \nu_{m,x} \) \( \mu \)-a.e.; and such that for all subsequences \( \{n'_{k,m}\} \subset \{n_{k,m}\} \), \( \{n'_{k,m}, x\} \) also has the asymptotic distribution \( \nu_{m,x} \) \( \mu \)-a.e. Note that \( \hat{\nu}_{m,x}(1) \neq 0 \) for \( x \in E_m \). By the lemma, we may choose \( \{\alpha_m\} \) such that \( \alpha_m > 0 \), \( \sum_{m \geq 1} \alpha_m = 1 \), and \( \sum_{m \geq 1} \alpha_m \hat{\nu}_{m,x}(1) \neq 0 \mu \)-a.e. Let \( \{n_{k,i,m}\}_{i=1}^{\infty} \) be any strictly increasing sequence such that for all \( m \),

\[
\lim_{I \to \infty} \frac{1}{I} \text{card}\{i \leq I: m_i = m\} = \alpha_m.
\]

Then it is easy to see by Weyl's criterion that \( \{n_{k,m}, x\} \) has the asymptotic distribution \( \sum \alpha_m \nu_{m,x} \mu \)-a.e. with \( (\sum \alpha_m \nu_{m,x})^{-1}(1) \neq 0 \mu \)-a.e. That is, \( F = \{x: \{n_{k,m}, x\} \) has an asymptotic distribution \( \nu_x \) with \( \hat{\nu}_x(1) \neq 0 \} \) is a \( W_1 \)-set such that \( \mu(E \setminus F) = 0 \). \( \square \)

The extension to LCA groups is immediate, save for one subtlety. Namely, given a collection of sequences \( \{\gamma_{k,m}\}_{k \geq 1} \subset \hat{G} (m \geq 1) \) with \( \lim_{k \to \infty} \gamma_{k,m} = \infty \), we must be able to mix subsequences of them (in appropriate proportions) so as to obtain a sequence still tending to \( \infty \). This is achieved by an easy adaptation of the proof of Theorem 14 in [3].

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