A GENERALIZATION OF SMITH THEORY

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ABSTRACT. Using Bredon cohomology, new relations are obtained between the mod p Betti numbers of a finite G-CW complex and its singular subspace, where G is a finite p-group.

Let G be a p-group of finite order $p^e$ and let X be a finite dimensional G-CW complex such that $H^*(X)$ is finite dimensional, where cohomology is understood with mod p coefficients. Let $SX$ denote the subcomplex of singular points of X, that is, of points fixed by some $g \neq e$. Finally, let $FX = X/SX$; $FX$ is a based G-CW complex such that the action off the basepoint is free. We seek relations among the mod p Betti numbers

$$a_q = \dim \tilde{H}^q(FX/G), \quad b_q = \dim H^q(X), \quad c_q = \dim H^q(SX).$$

If G is cyclic of order $p$, so that $SX = X^G$, Floyd's formulation [2, 4.4] of Smith theory gives the following inequality for $q \geq 0$ and $r \geq 0$, where $r$ is odd if $p$ is odd:

$$(*) \quad a_q + c_q + c_{q+1} + \cdots + c_{q+r} \leq b_q + b_{q+1} + \cdots + b_{q+r} + a_{q+r+1}.$$  

Floyd [2, p. 146] also gives the Euler characteristic relation

$$\chi(X) = \chi(X^G) + p \tilde{\chi}(FX/G),$$

where the reduced Euler characteristic of a based space is one less than the actual Euler characteristic. With $q = 0$ and $r$ large, $(*)$ gives $\sum c_q \leq \sum b_q$. When X is a mod p cohomology sphere, the last inequality and the relation $\chi(X) \equiv \chi(X^G) \mod p$ immediately imply Smith's conclusion that $X^G$ is also a mod p cohomology sphere.

In the general case, classical Smith theory and induction on $e$ imply dimensional restrictions on the cohomology of all fixed point spaces $X^H$ and therefore, by inductive use of Mayer-Vietoris sequences, on the cohomology of $SX$. Our new observation is that much sharper dimensional restrictions can be derived directly.

THEOREM. The following inequality holds for any $q \geq 0$ and $r \geq 0$:

$$a_q + \sum_{i=0}^{r} (p^e - 1)^i c_{q+i} \leq \sum_{i=0}^{r} (p^e - 1)^i b_{q+i} + (p^e - 1)^{r+1} a_{q+r+1}.$$  

In particular, with $r$ large,

$$\sum_{i \geq 0} (p^e - 1)^i c_{q+i} \leq \sum_{i \geq 0} (p^e - 1)^i b_{q+i}.$$
Moreover,
\[ \chi(X) = \chi(SX) + p^e \tilde{\chi}(FX/G). \]

Since \( a_q = 0 \) for \( q \) large by the finite dimensionality of \( X \), the inequalities and the finiteness of the \( b_q \) imply the finiteness of the \( a_q \) and \( c_q \). Of course, the Euler characteristic formula is trivial when \( X \) is a finite \( G \)-CW complex. If \( p = 2 \), the inequalities in the classical case \( e = 1 \) are those given by Floyd. If \( p > 2 \), the inequalities in the case \( e = 1 \) differ from those of Floyd due to the coefficients \( (p-1)^i \). We shall explain after the proof why the cyclic groups of odd prime order behave exceptionally.

When \( G \) is cyclic, our inequalities do not appear to give new information; in the noncyclic case, they do. For example, our inequalities obviously imply that if \( c_q = b_q \) for all \( q > n \), then \( c_n \leq b_n \). Even this simple fact does not seem to follow from any previous version of Smith theory.

To prove the theorem, observe first that the inequalities for \( r > 0 \) will follow inductively from those for \( r = 0 \), which read

(\#) \[ a_q + c_q \leq b_q + (p^e - 1)a_{q+1}. \]

To obtain the inequalities for \( r = 1 \), we add \((p^e - 1)c_{q+1}\) to both sides, and so on.

The proof of the theorem is an application of Bredon cohomology \([1]\). Let \( GO \) denote the category of orbits \( G/H \) and \( G \)-maps between them. A coefficient system is a contravariant functor from \( GO \) to the category of Abelian groups. For each coefficient system \( M \), there is a cohomology theory \( H^q_G(\cdot; M) \) on \( G \)-CW complexes. It is characterized by a dimension axiom: when restricted to the category \( GO \), \( H^q_G(\cdot; M) \) is the coefficient system \( M \) if \( q = 0 \) and is identically zero if \( q \neq 0 \). An exact sequence of coefficient systems \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) gives rise to a natural long exact sequence

\[ \cdots \rightarrow H^q_G(X; L) \rightarrow H^q_G(X; M) \rightarrow H^q_G(X; N) \rightarrow H^{q+1}_G(X; L) \rightarrow \cdots. \]

There are coefficient systems \( L, M, \) and \( N \) such that

\[ H^q_G(X; L) \cong \tilde{H}^q(FX/G), \quad H^q_G(X; M) \cong H^q(X), \quad \text{and} \quad H^q_G(X; N) \cong H^q(SX). \]

In fact, to specify \( L, M, \) and \( N \), we can and must set

\[ L(\cdot) = \tilde{H}^0(F?/G), \quad M(\cdot) = H^0(\cdot), \quad \text{and} \quad N(\cdot) = H^0(S?) \]
on orbits and on \( G \)-maps between orbits. Thus \( L(G) = Z_p \) and \( L(G/H) = 0 \) if \( H \neq e \); \( M(G/H) = Z_p[G/H] \) for all \( H \); and \( N(G) = 0 \) and \( N(G/H) = Z_p[G/H] \) if \( H \neq e \). In particular, \( M(G) \) is the group ring \( Z_p[G] \) regarded as a \( G \)-module.

Let \( I \) be the augmentation ideal in \( Z_p[G] \), let \( s \) be maximal such that \( I^s \neq 0 \), and let \( d_n \) be the dimension of the \( Z_p \)-vector space \( I^n/I^{n+1} \) for \( 1 \leq n \leq s \). The values of the \( d_n \) are given by Jennings' formula \([3, 2.10]\), but we only need the relations \( d_s = 1 \) and \( \sum d_n = p^e - 1 \). Write \( I^n \) ambiguously for both the ideal and the coefficient system with \( I^n(G) = I^n \) and \( I^n(G/H) = 0 \) for \( H \neq e \). Then \( I \) is a subcoefficient system of \( M \), and \( M/I = L \oplus N \) since \( Z_p[G]/I \cong Z_p \) and since the map

\[ Z_p[G/H] = H^0(G/H) \rightarrow H^0(G) = Z_p[G] \]

induced by a $G$-map $G \to G/H$ with $H \neq G$ takes values in $I$. The long exact cohomology sequence associated to the short exact sequence

$$0 \to I \to M \to L \oplus N \to 0$$

gives the inequality

$$a_q + c_q \leq b_q + \dim H_{G}^{q+1}(X; I)$$

and the Euler characteristic formula

$$\chi(X) = \chi(SX) + \chi(FX/G) + \chi(H_{G}^{*}(X; I)).$$

For $1 \leq n < s$, we have an evident short exact sequence

$$0 \to I^{n+1} \to I^n \to d_n L \to 0,$$

where $dL$ denotes the direct sum of $d$ copies of $L$. The resulting long exact sequence in cohomology gives

$$\dim H_{G}^{q}(X; I^n) \leq d_n a_q + \dim H_{G}^{q}(X; I^{n+1})$$

and

$$\chi(H_{G}^{q}(X; I^n)) = d_n \chi(FX/G) + \chi(H_{G}^{*}(X; I^{n+1})).$$

Since $d_s = 1$, $I^s = L$ and $H_{G}^{q}(X; I^s) = H^{q}(FX/G)$. Our theorem follows.

If $G$ is cyclic of odd prime order $p$, then $s = p - 1$ and $I^{p-1} = L$. Here $Z_p[G]/I^{p-1} \cong I$ as $Z_p[G]$-modules. If $t$ generates $G$, then the norm $\sum t^i$ generates $I^{p-1}$, and it follows that $M/I^{p-1} \cong I \oplus N$ as coefficient systems. We thus have short exact sequences

$$0 \to I \to M \to L \oplus N \to 0 \quad \text{and} \quad 0 \to L \to M \to I \oplus N \to 0.$$ 

With $\bar{a}_q = \dim H_{G}^{q}(X; I)$, these imply the two inequalities

$$a_q + c_q \leq b_q + \bar{a}_{q+1} \quad \text{and} \quad \bar{a}_q + c_q \leq b_q + a_{q+1}.$$ 

Floyd’s inequalities (*) follow inductively.

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BIBLIOGRAPHY