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RELATION BETWEEN RIGHT AND LEFT INVOLUTIONS OF A HILBERT ALGEBRA

P. P. SAWOROTNOW

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ABSTRACT. Existence of a densely defined right involution in a Hilbert algebra implies existence of a left involution.

H^*-algebras were introduced by W. Ambrose [**1**] to characterize Hilbert-Schmidt operators. This notion was generalized by M. F. Smiley [**7**], who showed that the structure theorems are valid also for a right H^*-algebra (a Hilbert algebra whose involution $x \to x^r$ satisfies the condition "$(yx, z) = (y, zx^r)$" but not the condition "$(xy, z) = (y, x^r z)$"). (Hilbert algebra here is a Banach algebra with a Hilbert space norm.) The author showed in [**5**, Theorem 2] that a proper right H^*-algebra is also a left H^*-algebra, i.e. it also has another involution $x \to x^l$ (a *left* involution) which satisfied the condition "$(xy, z) = (y, x^l z)$."

In this paper we shall show that the same is true also for the case when the involution $x \to x^r$ (the *right* involution) is defined on a dense subset only.

DEFINITION. Let A be a Hilbert algebra (A is a Banach algebra with a scalar product $(\ ,\ )$ such that $(x, x) = \|x\|^2$). We shall say that A is a *weak* right H^*-algebra if there is a dense subset D_r of A with the property that for each $x \in D_r$ there is some x^r such that $(yx, z) = (y, zx^r)$ for all $y, z \in A$. We define *weak* left H^*-algebra in a similar fashion.

The algebra A is said to be proper if each x^r is unique, i.e. A has a right involution $x \to x^r$, defined on a dense subset. Note that A is proper if $r(A) = \{u \in A: Au = (0)\}$ consists of zero alone. Algebra A in Example 2, p. 54, of [**5**] is an example of weak right (as well as left) H^*-algebra. Also it is easy to show that each weak right H^*-algebra is a right complemented algebra.

THEOREM. *Each proper weak right H^*-algebra A is a proper weak left H^*-algebra.*

PROOF. Note that the involution $x \to x^r$ of A, defined on a dense subset D_r, is closable, i.e. the closure of its graph is a graph of some mapping: it is easy to show that if $x_n \in D^r$, $x_n \to 0$ and $x_n^r \to y \in A$, then $y = 0$ (see end of section 8 in §5, Chapter I of [**3**]).

Now replace the scalar product $(\ ,\ )$ of A and the multiplication λx of members x of A and complex number λ by $[\ ,\ ]$ and $\lambda \circ x$ respectively, where these new

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products are defined as follows:

$$[x, y] = (y, x), \quad x, y \in A,$$
$$\lambda \circ x = \overline{\lambda} x, \quad \lambda \text{ is a complex number.}$$

Let A' denote the algebra consisting of members of A but with these new operations. Note that A' is also a weak right H^*-algebra.

Let $T': A \to A$ be a mapping defined by $T'x = x^r$ and let T be the closed extension of T' (we remarked above that the map $x \to x^r$ is closable). It follows from II in section 9 of §5 in Chapter I of [**3**], that T has an adjoint T^* defined on a dense subset D_l of A'. We define $x^l = T^*x$ for each $x \in D_l$. It follows that $(x^l, y) = (T^*x, y) = [x, Ty] = (y^r, x)$ for all $y \in D_r$, from which we see that $(x^l y, z) = (x^l, zy^r) = ((zy^r)^r, x) = (yz^r, x) = (y, xz)$ for all $y, z \in D_r$ (and each $x \in D_l$).

We leave it to the reader to deduce that $(xy, z) = (y, x^l z)$ for all $y, z \in A$.

To conclude the paper it is appropriate to comment that, in the presence of semisimplicity of the algebra A, the above Theorem follows from Theorem 2 of [**6**]. However, semisimplicity of a proper weak right H^*-algebra would be much more difficult to establish (if it is true at all) than in the case of a right H^*-algebra. In the latter case it follows from the fact that each ideal (whether closed or not) contains a (right) selfadjoint element (it is easy to show that a selfadjoint element is not a generalized nilpotent).

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DEPARTMENT OF MATHEMATICS, THE CATHOLIC UNIVERSITY OF AMERICA, WASHINGTON, D.C. 20064