UNIQUE SOLVABILITY OF AN EXTENDED STIELTJES MOMENT PROBLEM

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(Communicated by Paul S. Muhly)

ABSTRACT. Let \( a_1, \ldots, a_p \) be given real numbers ordered by size, and let \([\alpha, \beta]\) be a real interval disjoint from the set \( \{a_1, \ldots, a_p\} \). Let \( \{c_j^{(i)}: j = 1, 2, \ldots\} \), \( i = 1, \ldots, p \), be sequences of real numbers and \( c_0 \) be a real number. The extended Stieltjes moment problem is to find a distribution function \( \psi \) with all its points of increase in \([\alpha, \beta]\) such that

\[
\int_{\alpha}^{\beta} \frac{d\psi(t)}{(t-a_i)^j} = c_j^{(i)}, \quad i = 1, \ldots, p, \quad j = 1, 2, \ldots.
\]

Necessary and sufficient conditions for the existence of a unique solution of the problem are given. Orthogonal \( R \)-functions and Gaussian quadrature formulas play important roles in the proof.

1. Introduction. By a distribution function we shall mean a real-valued, bounded, nondecreasing function \( \psi(t) \) with an infinite number of points of increase. By its Stieltjes transform we mean the function \( \psi(z) = \int_{-\infty}^{\infty} d\psi(t)/(t-z) \). If all the points of increase of \( \psi \) lie in an interval \( I \), then \( \psi(z) \) is an analytic function in \( C - \overline{I} \).

Let \( a_1, a_2, \ldots, a_p \) be distinct real numbers, ordered by size. We shall call the real interval \([\alpha, \beta]\) a Stieltjes interval for the point set \( \{a_1, \ldots, a_p\} \) if \( (\alpha, \beta) \cap \{a_1, \ldots, a_p\} = \emptyset \). (Intervals of the form \((-\infty, 0]\) and \([0, \infty)\) are allowed.)

Let \( \{c_j^{(i)}: j = 1, 2, \ldots\} \), \( i = 1, \ldots, p \), be given sequences of real numbers and \( c_0 \) be a real number. The extended Stieltjes moment problem (ESMP) is defined as follows: Find a distribution function \( \psi(t) \) with all its points of increase in a given Stieltjes interval \([\alpha, \beta]\) such that

\[
\int_{\alpha}^{\beta} \psi(t) = c_0, \quad \int_{\alpha}^{\beta} \frac{d\psi(t)}{(t-a_i)^j} = c_j^{(i)}, \quad j = 1, 2, \ldots, i = 1, \ldots, p.
\]

In [3, 4] we studied the extended Hamburger moment problem (EHMP), which consists of finding a distribution function with the same properties as above, except the requirements that the points of increase lie in a given interval. Conditions for existence of a solution of the ESMP were briefly discussed in [5]. In this paper we give necessary and sufficient conditions for the existence of a unique solution of the problem. Since the tools used in proving the existence of solutions also are involved in the discussion of the uniqueness question, we give a very brief sketch of the existence proof here.

Received by the editors September 8, 1986.

1980 Mathematics Subject Classification (1985 Revision). Primary 30E05; Secondary 42C05.

Key words and phrases. Moment problems, orthogonal \( R \)-functions.
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The treatment of the uniqueness problem is related to the treatment of the uniqueness problem for the strong Stieltjes moment problem, given in terms of continued fractions in [1, 2]. For the classical Stieltjes moment problem, see e.g. [6, 7].

2. Orthogonal \( R \)-functions. Let \( \mathcal{R} \) denote the linear space consisting of all functions of the form

\[
R(z) = \alpha_0 + \sum_{i=1}^{p} \sum_{j=1}^{N_i} \frac{\alpha_{ij}}{(z - a_i)^j}, \quad \alpha_0, \alpha_{ij} \in \mathbb{C}.
\]

Elements of \( \mathcal{R} \) are called \( R \)-functions. We denote by \( \mathcal{R_R} \) the real space of all \( R \)-functions with real coefficients. A function \( R \) belongs to \( \mathcal{R} \) iff it can be written in the form \( R(z) = P(z)/Q(z) \), where \( Q \) is a polynomial with all its zeros among the points \( a_1, \ldots, a_p \), and where \( P \) is a polynomial with \( \deg P \leq \deg Q \). The spaces \( \mathcal{R} \) and \( \mathcal{R_R} \) are closed under multiplication.

Every natural number \( n \) has a unique decomposition \( n = qnp + r_n, \) \( 1 \leq r_n \leq p \). We write \( q = q_n, \) \( r = r_n \). We denote by \( \mathcal{R}_n \) the space of \( R \)-functions of the form

\[
R(z) = \alpha_0 + \sum_{i=1}^{p} \sum_{j=1}^{N_i} \frac{\alpha_{ij}}{(z - a_i)^j} + \sum_{i=r+1}^{p} \sum_{j=1}^{q} \frac{\alpha_{ij}}{(z - a_i)^j}.
\]

Let \( \Phi \) denote the linear functional defined on \( \mathcal{R} \) by

\[
\Phi \left( \alpha_0 + \sum_{i=1}^{p} \sum_{j=1}^{N_i} \frac{\alpha_{ij}}{(z - a_i)^j} \right) = \alpha_0 c_0 + \sum_{i=1}^{p} \sum_{j=1}^{N_i} \alpha_{ij} c_j^{(i)}.
\]

We shall say that \( \Phi \) is positive on an interval \([a, b]\) if \( \Phi(R) > 0 \) for every \( R \in \mathcal{R} \) such that \( R(t) \geq 0 \) and \( R(t) \neq 0 \) for \( t \in (a, b) \). It follows immediately that a necessary condition for the ESMP for a given Stieltjes interval \([\alpha, \beta]\) to have a solution is that \( \Phi \) is positive on \([\alpha, \beta]\). We shall in the following assume that this condition is satisfied.

If the functional \( \Phi \) is positive on some interval \([\alpha, \beta]\), it is positive on \((-\infty, \infty)\), and therefore gives rise to an inner product \( \langle , \rangle \) defined by \( \langle R, S \rangle = \Phi(R \cdot S) \). By applying the Gram-Schmidt procedure to the sequence

\[
\left\{ 1, \frac{1}{(z - a_1)}, \frac{1}{(z - a_1)^2}, \ldots, \frac{1}{(z - a_1)^2}, \ldots \right\}
\]

we obtain an orthonormal sequence \( \{Q_n\} \) of \( R \)-functions. We note that \( \langle Q_n, R \rangle = 0 \) for every \( R \in \mathcal{R}_{n-1} \). Furthermore \( Q_n \) may be written as \( Q_n(z) = V_n(z)/N_n(z), \) where

\[
N_n(z) = (z - a_1)^{q+1} \cdots (z - a_r)^{q+1} (z - a_{r+1})^{q} \cdots (z - a_p)^{q},
\]

and \( V_n(z) \) is a polynomial of degree at most \( n \).

3. Existence of solutions. When \( \Phi \) is positive on a Stieltjes interval \([\alpha, \beta]\), the polynomial \( V_n(z) \) is of degree \( n \), and has \( n \) real simple zeros in \((\alpha, \beta)\) (cf. [5], see also [3]). This can be seen as follows: Let \( t_1, \ldots, t_\lambda \) be all the zeros of odd order of \( V_n(z) \) in \((\alpha, \beta)\). If \( \lambda < n \), then the function

\[
T_n(z) = \frac{(z - t_1) \cdots (z - t_\lambda)(z - a_r)}{N_n(z)}
\]
belongs to \( \mathcal{R}_{n-1} \), hence \( \Phi(Q_nT_n) = 0 \). On the other hand the function

\[
T_n(z)Q_n(z) = \frac{(z - t_1) \cdots (z - t_n) \nu_n(z)(z - \alpha_r)}{N_n(z)^2}
\]

has a fixed sign in \((\alpha, \beta)\), since \((z - \alpha_r)\) does not change sign in \((\alpha, \beta)\). Thus \( \Phi(Q_nT_n) \neq 0 \), which is a contradiction. Consequently \( \lambda = n \), and the result follows.

With the zeros \( t_1^{(n)}, \ldots, t_n^{(n)} \) of \( \nu_n(z) \) (or equivalently of \( Q_n(z) \)) there is associated a Gaussian quadrature formula. There exist positive weights \( \lambda_n, 1, \ldots, \lambda_{n,n} \) such that

\[
\Phi(R) = \sum_{k=1}^{n} \lambda_{n,k} R(t_k^{(n)})
\]

for every \( R \) of the form \((2.1)\), where \( N_i \leq 2q + 2 \) for \( i < r \), \( N_i \leq 2q \) for \( i > r \), and \( N_r \leq 2q + 1 \) (see \[3, Theorem 4.2, Proposition 4.4; 4, Theorems 2.6, 2.7\]). For given \( i, j \) we may then write

\[
\phi_j^{(i)} = \Phi \left( \frac{1}{(z - \alpha_i)^j} \right) = \sum_{k=1}^{n} \lambda_{n,k} \frac{1}{(t_k^{(n)} - \alpha_i)^j}
\]

for sufficiently large \( n \). Thus we may also write \( \phi_j^{(i)} = \int_{\alpha}^{\beta} d\phi_n(t)/(t - \alpha_i)^j \), where the nondecreasing, uniformly bounded functions \( \phi_n(t) \) are defined by \( \phi_n(t) = \sum \{ \lambda_{n,k} : t_k^{(n)} \leq t \} \). Application of Helly’s selection and convergence theorems show that the sequence \( \{ \phi_n(t) \} \) contains at least one subsequence converging to a distribution function \( \phi \), and the limit function \( \phi \) of every such convergent subsequence has all its points of increase in \((\alpha, \beta)\) and satisfies

\[
\int_{\alpha}^{\beta} d\phi(t) = c_0, \quad \int_{\alpha}^{\beta} \frac{d\phi(t)}{t - \alpha_i)^j} = c_j^{(i)}, \quad j = 1, 2, \ldots, i = 1, \ldots, p.
\]

(For more details, see \[3\].) Thus we have seen

**Theorem 1.** A necessary and sufficient condition for the ESMP on the given Stieltjes interval \([\alpha, \beta]\) to have a solution is that \( \Phi \) is positive on \([\alpha, \beta]\).

**4. Uniqueness of solution.** We define the \( R \)-function \( P_n \) associated with \( Q_n \) by

\[
P_n(z) = \Phi_t \left( \frac{Q_n(t) - Q_n(z)}{t - z} \right).
\]

(The subscript \( t \) indicates that the functional operates on its argument as a function of \( t \).) The function \( P_n(z) \) can be written in the form \( P_n(z) = U_n(z)/N_n(z) \) where \( U_n(z) \) is a polynomial of degree at most \( n - 1 \).

From the Gaussian quadrature formula it follows that we may write

\[
P_n(z) = -\sum_{k=1}^{n} \lambda_{n,k} \frac{Q_n(z)}{t_k^{(n)} - z} = -Q_n(z) \int_{\alpha}^{\beta} \frac{d\phi_n(t)}{t - z},
\]

which means \( P_n(z)/Q_n(z) = -\phi_n(z) \).

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If the ESMP has a unique solution $\psi(t)$, then $\{\phi_n(t)\}$ converges to this solution, since otherwise convergent subsequences would produce distinct solutions. It follows that $\{\hat{\phi}_n(z)\}$ converges to $\hat{\psi}(z)$ for all $z \in \mathbb{C} - [\alpha, \beta]$. The sequence $\{P_n(z)/Q_n(z)\}$ is convergent for the same values of $z$.

We shall now show that if $\{P_n(z)/Q_n(z)\}$ converges on an interval of the form $(a_\rho, a_\sigma)$, where $(a_\rho, a_\sigma) \cap [\alpha, \beta] = \emptyset$, then the ESMP has a unique solution.

We note that the function
\[
\frac{1}{Q_n(t)} \left[ \frac{Q_n(t)}{N_{n-1}(t)} - \frac{Q_n(z)}{N_{n-1}(z)} \right]
\]
belongs to $\mathcal{R}_{n-1}$, and thus $\Phi(Q_n : f) = 0$. This implies
\[
\Phi_t \left( \frac{1}{t-z} \left[ \frac{Q_n(t)}{N_{n-1}(t)} - \frac{Q_n(z)}{N_{n-1}(z)} \right] \right) = \Phi_t \left( \frac{1}{t-z} \left[ \frac{Q_n(t) - Q_n(z)}{N_{n-1}(z)} \right] \right) + \Phi_t(Q_n(t)f(t))
\]
\[
= \frac{1}{N_{n-1}(z)} P_n(z).
\]
Hence
\[
P_n(z) = N_{n-1}(z) \Phi_t \left( \frac{1}{t-z} \left[ \frac{Q_n(t)}{N_{n-1}(t)} - \frac{Q_n(z)}{N_{n-1}(z)} \right] \right).
\]
We also note that the function
\[
g(t) = \frac{V_n(t) - V_n(z)}{(t-z)N_{n-1}(t)}
\]
belongs to $\mathcal{R}_{n-1}$, and thus
\[
\Phi \left( Q_n(t) \frac{V_n(t) - V_n(z)}{(t-z)N_{n-1}(t)} \right) = 0.
\]
Let $\phi$ be an arbitrary solution of the EHMP. From (4.3) we obtain
\[
P_n(z) = N_{n-1}(z) \int_{-\infty}^{\infty} \frac{Q_n(t)}{(t-z)N_{n-1}(t)} d\phi(t) - Q_n(z) \hat{\phi}(z).
\]
Similarly from (4.4) we get (noting that $N_n(t) = (t - a_r)N_{n-1}(t)$)
\[
\int_{-\infty}^{\infty} \frac{Q_n(t)}{(t-z)N_{n-1}(t)} d\phi(t) = \frac{1}{V_n(z)} \int_{-\infty}^{\infty} \frac{Q_n(t)^2(t-a_r)}{t-z} d\phi(t).
\]
Substitution of (4.6) into (4.5) and division by $Q_n(z)$ gives
\[
\frac{P_n(z)}{Q_n(z)} = \frac{1}{(z-a_r)Q_n(z)^2} \int_{-\infty}^{\infty} \frac{(t-a_r)Q_n(t)^2}{t-z} d\phi(t) - \hat{\phi}(z).
\]
Now assume that $\phi$ is a solution of the ESMP, so that we may write
\[
-\hat{\phi}(z) = \frac{P_n(z)}{Q_n(z)} - \frac{1}{(z-a_r)Q_n(z)^2} \int_{a_r}^{\beta} \frac{(t-a_r)Q_n(t)^2}{t-z} d\phi(t).
\]
Let \( z \in (a_\rho, a_\sigma) \) for some \( \rho, \sigma \), where \( (a_\rho, a_\sigma) \cap [\alpha, \beta] = \emptyset \). Then either \((t - a_\rho), (t - a_\sigma), \) and \((t - z)\) are positive for all \( t \in [\alpha, \beta] \), or \((t - a_\rho), (t - a_\sigma), \) and \((t - z)\) are negative for all \( t \in [\alpha, \beta] \). The numbers \((z - a_\rho)\) and \((z - a_\sigma)\) have opposite sign. Therefore \( \phi(z) + P_n(z)/Q_n(z) \) and \( \phi(z) + P_m(z)/Q_m(z) \) have opposite sign when \( n \) is of the form \( n = pk + \rho \) and \( m \) is of the form \( m = pk + \sigma \). It follows that if the sequences \( \{P_{pk+\rho}(z)/Q_{pk+\rho}(z): k = 0, 1, 2, \ldots\} \) and \( \{P_{pk+\sigma}(z)/Q_{pk+\sigma}(z): k = 0, 1, 2, \ldots\} \) are convergent and converge to the same value, then this value is \(-\phi(z)\). Thus if these two sequences converge to the same function on a subinterval of \((a_\rho, a_\sigma)\), then this function is \(-\phi(z)\). From the uniqueness of the Stieltjes transform of the solution follows uniqueness of the solution itself. (Recall that the transform is known on \( \mathbb{C} - [\alpha, \beta] \) when it is known on an interval.)

We may sum up the results in this section as follows:

**Theorem 2.** The following statements are equivalent.

(A) The ESMP has a unique solution.

(B) The sequence \( P_n(z)/Q_n(z) \) converges on \( \mathbb{C} - [\alpha, \beta] \).

(C) There exist an interval \((a_\rho, a_\sigma)\) disjoint from \([\alpha, \beta]\) such that the sequences \( \{P_{pk+\rho}(z)/Q_{pk+\rho}(z): k = 0, 1, 2, \ldots\} \) and \( \{P_{pk+\sigma}(z)/Q_{pk+\sigma}(z): k = 0, 1, 2, \ldots\} \) converge to the same value (depending on \( z \)) for every \( z \) in some interval \((a, b) \subset (a_\rho, a_\sigma)\).

**References**

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