HEREDITARY $C^*$-SUBALGEBRAS OF $C^*$-CROSSED PRODUCTS
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ABSTRACT. Let $(A,G,\alpha)$ be a $C^*$-dynamical system. Assume that $B$ is an $\alpha$-invariant $C^*$-subalgebra of $A$. Then we shall give a necessary and sufficient condition for $B \times_\alpha G$ to be a $C^*$-subalgebra of $A \times_\alpha G$, where $B \times_\alpha G$ (resp. $A \times_\alpha G$) denotes a $C^*$-crossed product of $B$ (resp. $A$) by a locally compact group $G$. Moreover, we shall show that if $B$ is an $\alpha$-invariant hereditary $C^*$-subalgebra of $A$, then $B \times_\alpha G$ is a hereditary $C^*$-subalgebra of $A \times_\alpha G$.

1. Introduction. Let $(A,G,\alpha)$ be a $C^*$-dynamical system, namely, a $C^*$-algebra $A$ and a homomorphism $\alpha$ from a locally compact group $G$ into the automorphism group of $A$ such that $G \ni t \rightarrow \alpha_t(x)$ is continuous for each $x$ in $A$. For a given $(A,G,\alpha)$, we can construct a new $C^*$-algebra, called the $C^*$-crossed product of $A$ by $G$ and denoted by $A \times_\alpha G$ (see [4] for the details). Let $B$ be an $\alpha$-invariant $C^*$-subalgebra of $A$. We very often require that $B \times_\alpha G$ is a $C^*$-subalgebra of $A \times_\alpha G$ in studying the $C^*$-crossed products or more widely objects in $C^*$-dynamical systems. If $G$ is amenable, $B \times_\alpha G$ is always a $C^*$-subalgebra of $A \times_\alpha G$ (see [4, 7.7.7 and 7.7.9]). But $B \times_\alpha G$ is not necessarily a $C^*$-subalgebra of $A \times_\alpha G$ if $G$ is not amenable. It is known that if $B$ is an $\alpha$-invariant ideal of $A$, then $B \times_\alpha G$ is a $C^*$-subalgebra of $A \times_\alpha G$ (see [2, Proposition 12]).

In §3, we shall give a necessary and sufficient condition for $B \times_\alpha G$ to be a $C^*$-subalgebra of $A \times_\alpha G$. Moreover, we shall show that if $B$ is an $\alpha$-invariant hereditary $C^*$-subalgebra of $A$, then $B \times_\alpha G$ is a hereditary $C^*$-subalgebra of $A \times_\alpha G$. Finally, we shall state an example where $B \times_\alpha G$ is not a $C^*$-subalgebra of $A \times_\alpha G$.

2. Preliminaries. Let $(A,G,\alpha)$ be a $C^*$-dynamical system. A $C^*$-crossed product $A \times_\alpha G$ for $(A,G,\alpha)$ is defined as the enveloping $C^*$-algebra of $L^1(A,G)$, the set of all Bochner integrable $A$-valued functions on $G$ equipped with the following Banach $*$-algebra structure:

$$(xy)(t) = \int_G x(s)\alpha_s(y(s^{-1}t))\, ds,$$

$$x^*(t) = \Delta(t)^{-1}\alpha_t(x(t^{-1}))^*,$$

$$\|x\|_1 = \int_G \|x(s)\|\, ds,$$

where $ds$ is the left Haar measure of $G$ and $\Delta(t)$ is the associated modular function on $G$.
Let $G \ni t \mapsto u_t$ be the canonical unitary representation of $G$ into the multiplier algebra $M(A \times_\alpha G)$ of $A \times_\alpha G$ with $\alpha_t(\cdot) = u_t^* u_t$. For each $\varphi$ in $(A \times_\alpha G)^*$, there corresponds a function $\Phi : G \to A^*$ given by

$$\Phi(t)(x) = \varphi(xu_t)$$

for all $t$ in $G$ and $x$ in $A$. The set of such functions is denoted by $B(A \times_\alpha G)$, and each element in $B(A \times_\alpha G)$ arising from a positive linear functional of $A \times_\alpha G$ is said to be positive definite with respect to $\alpha$ (Pedersen [4, 7.6.7]). Since we conversely see that

$$\varphi(y) = \int_G \Phi(t)(y(t)) \, dt$$

for $y$ in $L^1(A,G)$, the correspondence $\varphi \mapsto \Phi$ is a bijection from $(A \times_\alpha G)^*$ onto $B(A \times_\alpha G)$ (see [4, 7.6.7] for the details). Now we denote by $B_+(A \times_\alpha G)$ the set of positive definite functions with respect to $\alpha$ in $B(A \times_\alpha G)$.

3. Results.

**Theorem 1.** Let $(A,G,\alpha)$ be a $C^*$-dynamical system, and let $B$ be an $\alpha$-invariant $C^*$-subalgebra of $A$. Then the following statements (i), (ii) are equivalent.

(i) $B \times_\alpha G$ is a $C^*$-subalgebra of $A \times_\alpha G$.

(ii) For any $\varphi$ in $B_+(B \times_\alpha G)$, there exists a positive definite function $\Psi$ in $B_+(A \times_\alpha G)$ such that $\Psi(t)|_B = \Phi(t)$ for all $t$ in $G$ and $\|\Psi(e)\| = \|\Phi(e)\|$ for the identity $e$ of $G$.

**Proof.** (i) $\Rightarrow$ (ii). Identifying $A$ with its image in $M(A \times_\alpha G)$, we denote by $u$ the canonical unitary representation of $G$ into $M(A \times_\alpha G)$ satisfying $\alpha_t(a) = u_t a u_t^*$ for all $a$ in $A$. Identifying $B \times_\alpha G$ with the image under its universal representation and denoting by $\lambda$ the canonical unitary representation of $G$ into $M(B \times_\alpha G)$ satisfying $\alpha_t(b) = \lambda_t b \lambda_t^*$ for all $b$ in $B$, there exists an isomorphism $\rho$ from $B \times_\alpha G$ onto its image under the universal representation of $A \times_\alpha G$ such that $\rho(b) = b$ for all $b$ in $B$ and $\rho(\lambda_t) = u_t$ for all $t$ in $G$ (cf. [4, 7.6.6]).

Take a positive linear functional $\varphi$ of $B \times_\alpha G$ corresponding to $\Phi$. Then there exists a positive linear functional $\psi$ of $A \times_\alpha G$ such that $\psi|_{\rho(B \times_\alpha G)} = \varphi \circ \rho^{-1}$ and $\|\psi\| = \|\varphi\|$. We define a positive definite function $\Psi$ in $B_+(A \times_\alpha G)$ by

$$\Psi(t)(x) = \psi(xu_t)$$

for all $t$ in $G$ and $x$ in $A$. For $b$ in $B$, we then have

$$\Psi(t)(b) = \psi(bu_t) = \varphi(\rho^{-1}(bu_t)) = \varphi(b\lambda_t) = \Phi(t)(b).$$

For $x$ in $A$ and $f$ in $L^1(G)$, put $y(t) = f(t)x$, so $y$ in $L^1(A,G)$, which is identified with

$$y = \int_G x u_t f(t) \, dt.$$ 

Using the Cauchy-Schwarz inequality, we easily see that

$$|\psi(y)|^2 \leq \|\psi\|^2 \|f\|^2 \psi(xx^*).$$
When $x$ and $f$ range over an approximate identity for $A$ and an approximate identity for $L^1(G)$ respectively, it follows from [1, 2.1.5 and 2.7.5] that $\|\psi\| \leq \|\psi\|_A = \|\Psi(e)\|$. This means that $\|\psi\| = \|\Psi(e)\|$. Similarly, we see that $\|\varphi\| = \|\Phi(e)\|$. Thus we obtain $\|\Psi(e)\| = \|\Phi(e)\|$. 

(ii) $\Rightarrow$ (i). Take any positive linear functional $\varphi$ of $B \times_A G$ with $\|\varphi\| \leq 1$. We denote by $\Phi$ a positive definite function in $B_+(B \times_A G)$ corresponding to $\varphi$. By the assumption, we can choose a positive definite function $\Psi$ in $B_+(A \times_A G)$ satisfying $\Psi(t)|_B = \Phi(t)$ for all $t$ in $G$ and $\|\Psi(e)\| = \|\Phi(e)\|$. Then there corresponds a positive linear functional $\psi$ of $A \times_A G$ to $\Psi$. For any $x$ in $L^1(B,G)$, we denote by $\|x\|_{B \times_A G}$ (resp. $\|x\|_{A \times_A G}$) the $C^*$-norm of $x$ in $B \times_A G$ (resp. $A \times_A G$). In order to prove the statement (i), it suffices to show that $\|x\|_{B \times_A G} = \|x\|_{A \times_A G}$. Now we have 

$$\varphi(x^* x) = \int G \Phi(t)(x^* x(t))\,dt = \int G \Psi(t)(x^* x(t))\,dt = \psi(x^* x).$$

Since we have 

$$\|\varphi\| = \|\Phi(e)\| = \|\Psi(e)\| = \|\psi\|,$$

we see that $\|\psi\| \leq 1$. Thus we conclude that $\|x\|_{B \times_A G} \leq \|x\|_{A \times_A G}$. Since the reverse inequality is clear, we complete the proof. Q.E.D.

Let $(B, G, \alpha)$ be a $C^*$-dynamical system. If $\alpha_t^{**}$ denotes the double transpose of $\alpha_t$, then the map $G \ni t \rightarrow \alpha_t^{**}$ is a homomorphism of $G$ into the automorphism group of the enveloping von Neumann algebra $B^{**}$ of $B$.

**Lemma 2.** Let $(B, G, \alpha)$ be a $C^*$-dynamical system. Take any $\Phi$ from $B_+(B \times_A G)$. Then we have 

$$\sum_{ij} \langle \Phi(s_i^{-1}s_j), \alpha_{s_i^{-1}}^{**}(x_i^* x_j) \rangle \geq 0$$

for finite sets $\{s_i\}$ in $G$ and $\{x_i\}$ in $B^{**}$.

**Proof.** For $x_i$ in $B^{**}$, there exists a net $\{x_{i(k)}\}_k$ in $B$ with $\|x_{i(k)}\| \leq \|x_i\|$ for all $k$ such that the net $\{x_{i(k)}\}_k$ is $\sigma$-strongly* convergent to $x_i$ (cf. [5, II, Lemma 2.5 and Theorem 4.8]). Then we see that $\{x_{i(k)}^* x_j(k)\}_k$ is $\sigma$-weakly convergent to $x_i^* x_j$. Since $\Phi(s_i^{-1}s_j)$ is an element in $B^*$ and $\alpha_t^{**}$ is normal for all $t$ in $G$, using [4, 7.6.8] we have 

$$0 \leq \sum_{ij} \langle \Phi(s_i^{-1}s_j), \alpha_{s_i^{-1}}^{**}(x_i^* (k)x_j(k)) \rangle$$

$$= \sum_{ij} \langle \Phi(s_i^{-1}s_j), \alpha_{s_i^{-1}}^{**}(x_i^* (k)x_j(k)) \rangle$$

$$= \sum_{ij} \langle \Phi(s_i^{-1}s_j), \alpha_{s_i^{-1}}^{**}(x_i^* (k)x_j(k)) \rangle.$$

Consequently we obtain the desired result. Q.E.D.

**Lemma 3.** Let $(A, G, \alpha)$ be a $C^*$-dynamical system. Let $B$ be an $\alpha$-invariant hereditary $C^*$-subalgebra of $A$. Then $B \times_A G$ is a $C^*$-subalgebra of $A \times_A G$.

**Proof.** Let $E$ be a conditional expectation from $A^{**}$ onto $B^{**}$. $E$ may be given by $E(x) = pxp$ for all $x$ in $A^{**}$, where $p$ is an $\alpha^{**}$-invariant projection in $A^{**}$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
with $B^{**} = pA^{**}p$. First we remark that
\[ \alpha_t^{**}(E(x)) = E(\alpha_t(x)) \]
for all $x$ in $A$ and $t$ in $G$.

For any $\Phi$ in $B_+ (B \rtimes_{\alpha} G)$, we define a function $\Psi : G \to A^*$ by
\[ \Psi(t)(x) = (\Phi(t), E(x)) \]
for all $x$ in $A$ and $t$ in $G$. Take finite sets $\{s_i\}$ from $G$ and $\{x_i\}$ from $A$. Since $E$ is completely positive, we have
\[ E(x_i^* x_j) = \sum_k y_i(k) y_j(k) \]
for some $\{y_i(k)\}_{k} \in B^{**}$ (cf. [5, IV. Lemma 3.1]). Then we have
\[ \sum_{ij} \Psi(s_i^{-1}s_j)(\alpha_{s_i^{-1}}(x_i^* x_j)) = \sum_{ij} (\Phi(s_i^{-1}s_j), E(\alpha_{s_i^{-1}}(x_i^* x_j))) \]
\[ = \sum_{ij} (\Phi(s_i^{-1}s_j), \alpha_{s_i^{-1}}^{**}(E(x_i^* x_j))) \]
\[ = \sum_{ij} (\Phi(s_i^{-1}s_j), \sum_k \alpha_{s_i^{-1}}^{**}(y_i(k) y_j(k))) \]
\[ = \sum_k \sum_{ij} (\Phi(s_i^{-1}s_j), \alpha_{s_i^{-1}}^{**}(y_i(k) y_j(k))) \]
\[ \geq 0. \]

Here the last inequality follows from Lemma 2. Since $\Phi$ is bounded and norm continuous on $G$, it is easy to check the boundedness and norm continuity of $\Psi$ on $G$. Therefore it follows from [4, 7.6.8] that $\Psi$ is positive definite with respect to $\alpha$. Since the routine observations show that $\Psi(t)|_B = \Phi(t)$ for all $t$ in $G$ and $\|\Psi(e)\| = \|\Phi(e)\|$, we obtain the desired result from Theorem 1. Q.E.D.

**Theorem 4.** Let $(A, G, \alpha)$ be a $C^*$-dynamical system. Let $B$ be an $\alpha$-invariant hereditary $C^*$-subalgebra of $A$. Then $B \rtimes_{\alpha} G$ is a hereditary $C^*$-subalgebra of $A \rtimes_{\alpha} G$.

**Proof.** In order to prove our result, it suffices to show
\[ (B \times_{\alpha} G)^{**} = p (A \times_{\alpha} G)^{**} p \]
for some open projection $p$ in $(A \times_{\alpha} G)^{**}$.

Now we may write the universal representation of $A \times_{\alpha} G$ as the induced representation $(\pi \times u, H)$ via some covariant representation $(\pi, u, H)$ of $A$ (see [4, 7.6.4] for the notation of $(\pi \times u, H)$). Here we note that
\[ (\pi \times u)(A \times_{\alpha} G)^w = (A \times_{\alpha} G)^{**} \]
where $(\quad)^w$ denotes the weak closure of $(\quad)$. Since $B$ is a hereditary subalgebra $A$, we have $B^{**} = qA^{**}q$ for some open projection $q$ in $A^{**}$. We denote by $\pi^{**}$ the normal extension of $\pi$ from $A^{**}$ onto $\pi(A)^w$ and put $p = \pi^{**}(q)$. Then we easily see that
\[ \pi(B)^w = p \pi(A)^w p. \]
Since $\pi^{**}$ is normal, $p$ is a strong limit of a monotone increasing net of positive elements in $\pi(A)$. Hence, applying [4, 3.11.9 and 3.12.9], we easily see that $p$ is an open projection in $(A \times_\alpha G)^{**}$. Since $(\pi, u, H)$ is a covariant representation of $A$ and $q$ is $\alpha^{**}$-invariant, we see that $u_t p u_t^* = p$ for all $t$ in $G$. Hence, if we put
\[ u_f = \int_G f(t) u_t \, dt \]
for all $f$ in $L^1(G)$, we obtain
\[ p u_f = u f p. \]
Since $(\pi \times u)(A \times_\alpha G)^w$ (resp. $(\pi \times u)(B \times_\alpha G)^w$) is generated by $\{\pi(x) u_f | x \in A, f \in L^1(G)\}$ (resp. $\{\pi(x) u_f | x \in B, f \in L^1(G)\}$), the formula
\[ p \pi(x) u_f = p \pi(x) p u_f \]
for any $x$ in $A$ shows
\[ p(\pi \times u)(A \times_\alpha G)^w = (\pi \times u)(B \times_\alpha G)^w = (B \times_\alpha G)^{**}. \]
Thus we complete the proof. Q.E.D.

REMARK 5. As an alternative proof of the above theorem, it is also possible that we directly show by a few computations that $B \times_\alpha G$ is a closed linear span of $(B \times_\alpha G)(A \times_\alpha G)(B \times_\alpha G)$.

We end this paper by stating an example where $B \times_\alpha G$ is not a $C^*$-subalgebra of $A \times_\alpha G$.

EXAMPLE 6. Let $G$ be a locally compact group whose enveloping group $C^*$-algebra $C^*(G)$ is not nuclear. Hence, it follows from [3, Theorem A] that there exist a $C^*$-algebra $A$ and a $C^*$-subalgebra $B$ of $A$ such that the projective $C^*$-tensor product $B \otimes_{\max} C^*(G)$ can not be embedded in the projective $C^*$-tensor product $A \otimes_{\max} C^*(G)$. Consider a $C^*$-dynamical system $(A, G, \alpha)$, where $\alpha$ is the trivial action on $G$. Then $A \times_\alpha G$ and $B \times_\alpha G$ are nothing but $A \otimes_{\max} C^*(G)$ and $B \otimes_{\max} C^*(G)$, respectively.

Note also that when we consider a suitable nonamenable discrete group such as the free group on two generators, it is possible to find a nontrivial action for which the above is valid.

For the theory of $C^*$-tensor products of $C^*$-algebras, the reader is referred to [3, or 5].

REFERENCES