RANDOM SIGN EMBEDDINGS FROM $l^n_r$, $2 < r < \infty$

T. FIGIEL, W. B. JOHNSON, AND G. SCHECHTMAN

(Communicated by William J. Davis)

ABSTRACT. Estimates for any ideal norm of a “random sign embedding” from $l^n_r$ into $l^n_p$, $2 < r < \infty$, are given in terms of the corresponding ideal norm of the identity of $l^n_k$, $k = k(n, m, r)$.

1. Introduction. A norm one linear operator $u$ from $l^n_r$ ($1 < r < \infty$) into a Banach space is called a $\delta$-sign embedding (where $\delta > 0$) provided

$$
\left\| u \sum_{i=1}^{n} \varepsilon_i e_i \right\| \geq \delta n^{1/r}
$$

for all choices of signs $\varepsilon_i \pm 1$. (Here $(e_i)_{i=1}^{n}$ denotes the unit vector basis for $l^n_r$.) If the weaker condition

$$\text{Average} \left\{ \left\| u \sum_{i=1}^{n} \varepsilon_i e_i \right\| : (\varepsilon_i) \in \{-1, 1\}^n \right\} \geq \delta n^{1/r}
$$

is satisfied, we say that $u$ is a random $\delta$-sign embedding.

The most striking result concerning these concepts is J. Elton's theorem [E] (extended by A. Pajor [Pa] to complex Banach spaces) that if $u$ is a random $\delta$-sign embedding from $l^n_1$, then there is a subset $A$ of $\{1, \ldots, n\}$ with cardinality $k \equiv |A| \geq \varepsilon n$ so that the restriction of $u$ to $l^n_A$ has isomorphism constant at most $K$, where $\varepsilon > 0$ and $K < \infty$ are constants which depend only on $\delta$. Predating Elton’s theorem was the result from [JS] that when the random $\delta$-sign embedding $u$ from $l^n_1$ takes values in $L_1$, the space $u l^n_A$ is a well-complemented copy of $l^n_1$ in $L_1$ and hence the identity operator on $l^n_1$ well factors through the operator $u$. From results stated but not proved in [JS] it can be derived that if $u$ is a random $\delta$-sign embedding from $l^n_1$ into $L_p$ for $1 < p < 2$, then the identity on $l^n_k$ well factors through $u$ for some $k \geq n^{1-\varepsilon}$ and any $\varepsilon > 0$ (where, of course, how well depends on $\varepsilon$). Actually, simpler considerations yield that the identity on $l^n_k$ with $k$ proportional to $n$ weakly factorizes through such a $u$; this fact is recorded in Remark 3. (“Weak factorization” is defined in Proposition 2.)

There cannot be any such results in the range $r > 2$ because

$$n^{1/r-1/2} I : l^n_r \to l^n_2
$$

is a 1-sign embedding and hence by the results of [FLM or BDGJN], there is for some $\delta > 0$ a $\delta$-sign embedding $u$ from $l^n_r$ into $l^n_m$ with $m \leq n^{r/2}$ whose factorization
constant \( \gamma_2(u) \) through a Hilbert space is less than 2. The purpose of this note is to observe that this phenomenon cannot occur if \( m \) is proportional to \( n \); for example, if \( 2 < r < \infty \) and \( u: l_r^n \rightarrow l_2^m \) is a random \( \delta \)-sign embedding, then the identity operator on \( l_r^k \) with \( k \) proportional to \( n \) has a good weak factorization through \( u \), which yields that for any ideal norm \( \alpha \),

\[
\alpha(u) \geq \varepsilon \alpha(l_r^n),
\]

where \( \varepsilon > 0 \) is a constant which depends only on \( \delta \) and \( r \) and

\[
\alpha(l_r^n) \equiv \alpha(I: l_r^n \rightarrow l_r^n).
\]

We are interested in this result mainly for the ideal norm \( S_p \), \( 1 \leq p < \infty \), defined by

\[
S_p(u: X \rightarrow Y) = \inf \left\{ \|w\|\|v\|: u = vw, \ w: X \rightarrow E, \ v: E \rightarrow Y, \ E \text{ a subspace of } L_p(\mu) \text{ for some measure } \mu \right\}.
\]

This case is applied in [CJ]; indeed, the possibility of such an application motivated this note. In fact, in the first version of this note, Theorem 1 was stated only for \( \alpha = S_p \). We are indebted to the referee for pointing out that our argument yields Proposition 2 and the present statement of Theorem 1.

In §2 we repeat the known observation that, up to constants depending on \( p \) and \( r \), if \( r > 2 \) and \( 1 < p < \infty \), then \( S_p(l_r^n) \) is the same as \( \gamma_p(l_r^n) \), the factorization constant of the identity of \( l_r^n \) through an \( L_p \) space. This latter constant is known, up to constants depending on \( r \) and \( p \), to be of order \( n^a \) for an appropriate exponent \( \alpha = \alpha(p, r) \). The values for \( \alpha \) are catalogued in Pietsch’s book [P] and repeated in §2.

We use standard Banach space theory notation and terminology as may be found in [LT].

2. Random sign embeddings from \( l_r^n \). We now state the result of this note.

THEOREM 1. Let \( 2 < r < \infty \). If \( u: l_r^n \rightarrow l_r^n \) is a random \( \delta \)-sign embedding, then for any ideal norm \( \alpha \),

\[
\alpha(u) \geq \tau \alpha(l_r^n),
\]

where

\[
\tau = 2^{-1/\tau} (\delta/r^{1/2})^{2(r-1)/(r-2)} (n/m)^{1/(r-2)}
\]

and

\[
k \geq \tau^r n/(1 - \tau^r).
\]

As pointed out by the referee, Theorem 1 is an immediate consequence of Proposition 2. The referee’s concept of weak factorization introduced in the proposition is the obvious operator theoretic analogue to the weak distance studied by N. Tomczak-Jaegermann [T-J].

PROPOSITION 2. Let \( r, u, \delta, \tau \) and \( k \) be as in Theorem 1. Then \( I: l_r^k \rightarrow l_r^k \) \( \tau^{-1} \)-weakly factorizes through \( u \); i.e., \( I \) is in the convex hull of the set

\[
\{wuv: v: l_r^k \rightarrow l_r^n, w: l_r^n \rightarrow l_r^k; \|w\| \cdot \|v\| \leq \tau^{-1}\}.
\]
PROOF. From Khintchine's inequality we have

\[ \left\| \left( \sum_{i=1}^{n} |u \cdot e_i|^2 \right)^{1/2} \right\|_r \geq \delta r^{-1/2} n^{1/r}, \]

where the functional operations \(|x|^2\) and \(|x|^{1/2}\) are taken with respect to the lattice structure of \(l^m\).

On the other hand, regarding \(u\) as an operator into \(l^\infty\),
\[ \|u\|_{l^\infty} \leq \|u\|_{l^r} \rightarrow l^m \| = 1, \]
so, setting \(1/r + 1/s = 1\), we easily check that

\[ \left\| \left( \sum_{i=1}^{n} |u \cdot e_i|^s \right)^{1/s} \right\|_\infty \leq 1. \]

Indeed,
\[ \left\| \left( \sum_{i=1}^{n} |u \cdot e_i|^s \right)^{1/s} \right\|_\infty = \max_{1 \leq j \leq m} \sup_{1 \leq i \leq n} \left\{ \left( \sum_{i=1}^{n} |a_i u \cdot e_i|^s \right)^{1/s} : \left( \sum_{i=1}^{n} |a_i|^r \right)^{1/r} = 1 \right\} \]
\[ = \sup \left\{ \left\| u \sum_{i=1}^{n} a_i e_i \right\|_\infty : \left( \sum_{i=1}^{n} |a_i|^r \right)^{1/r} = 1 \right\} \]
\[ = \|u\|_{l^r} \rightarrow l^m \| \leq 1. \]

An extrapolation argument now yields

\[ \max_{1 \leq i \leq n} |u \cdot e_i| \geq 2^{1/r} r n^{1/r}. \]

To see this, set \(\theta = s/2\), so that \(1/2 = \theta/s + (1 - \theta)/\infty\). Then

\[ \delta r^{-1/2} n^{1/r} \leq \left\| \left( \sum_{i=1}^{n} |u \cdot e_i|^2 \right)^{1/2} \right\|_r \text{ (by (i))} \]
\[ \leq \left\| \left( \sum_{i=1}^{n} |u \cdot e_i|^s \right)^{\theta/s} \max_{1 \leq i \leq n} |u \cdot e_i|^{1-\theta} \right\|_r \]
\[ \leq \left\| \left( \sum_{i=1}^{n} |u \cdot e_i|^s \right)^{1/s} \right\|_r \max_{1 \leq i \leq n} |u \cdot e_i|^{1-\theta} \]
\[ \leq m^{\theta/r} \left\| \left( \sum_{i=1}^{n} |u \cdot e_i|^s \right)^{1/s} \right\|_r \max_{1 \leq i \leq n} |u \cdot e_i|^{1-\theta} \]
\[ \leq m^{\theta/r} \left\| \max_{1 \leq i \leq n} |u \cdot e_i|^{1-\theta} \right\|_r \text{ (by (ii))} . \]

The conclusion of Proposition 2 follows in a formal way from condition (iii) via a combination of more-or-less standard techniques.
For $1 \leq i \leq n$, define

$$A_i' = \left\{ j : |ue_i(j)| = \max_{1 \leq k \leq m} |ue_i(k)| \right\}$$

and set $A_i = A_i' \setminus \bigcup_{j<i} A_j'$. Since the $A_i$'s are pairwise disjoint, we have

$$(iv) \quad \left( \sum_{i=1}^{n} ||ue_i 1_{A_i}||_r \right)^{1/r} = \left( \sum_{i=1}^{n} ||ue_i||_r \right) = \max_{1 \leq i \leq r} ||ue_i||_r.$$ 

Setting $A = \{ i \leq n : ||ue_i 1_{A_i}|| \geq \tau \}$, we check that

$$(v) \quad |A| \equiv k \geq \tau^n n/(1 - \tau^r).$$

Indeed, since $||ue_i 1_{A_i}||_r \leq 1$ for each $i$, (iv) and (iii) yield $k + \tau(n - k) \geq 2\tau n$ so that $(1 - \tau^r)k \geq \tau^n$.

The rest of the proof is really just an application of Tong's diagonal principle (see [LT, p. 20]).

Assume, for notional convenience, that $A = \{1, \ldots, k\}$ and let $P$ be the norm one projection from $l^m_r$ onto $\text{span}(ue_i 1_{A_i})_{i=1}^k$. For each $\epsilon$ in $\{-1, 1\}^k$, define $v_{\epsilon} : l^k_r \to l^m_r$ and $\tilde{w}_{\epsilon} : \text{span}(ue_i 1_{A_i})_{i=1}^k \to l^k_r$ by setting

$$v_{\epsilon} e_i = \epsilon(i) e_i; \quad \tilde{w}_{\epsilon}(ue_i 1_{A_i}) = \epsilon(i) e_i$$

and let $w_{\epsilon} = \tilde{w}_{\epsilon} P$; $w_{\epsilon} : l^m_r \to l^k_r$. Evidently $||v_{\epsilon}|| = 1$, $||w_{\epsilon}|| \leq \tau^{-1}$, and it is easy to check that $I : l^k_r \to l^k_r = \text{Average}_{\epsilon} w_{\epsilon} w_{\epsilon}$. $\square$

REMARK 3. Suppose that $u : l^k_r \to L_r$ is a random $\delta$-sign embedding and $1 < r < 2$. Khintchine's inequality yields that

$$\left(\frac{\delta}{2}\right)n^{1/r} \leq \left( \sum_{i=1}^{n} |ue_i|^{r} \right)^{1/2} \leq \left( \sum_{i=1}^{n} |ue_i|^{r} \right)^{1/r} \leq n^{1/r},$$

so the extrapolation argument used in the proof of Proposition 2 gives

$$\left(\frac{\delta}{2}\right)^{2/(2-r)} n^{1/r} \leq \max_{1 \leq i \leq n} |ue_i|.$$

The further argument in Proposition 2 shows that the identity operator on $l^k_r$ with $k$ proportional to $n$ has a good weak factorization through $u$.

We close with a catalogue of estimates for $S_p(l^k_r)$ when $2 < r < \infty$. First, a simple exchangeability argument (see, for example, [JMST, p. 34]) shows that $S_p(l^k_r)$ is obtained via an operator which maps $(e_i)_{i=1}^k$ onto a normalized 1-symmetric basic sequence in $L_p$. Now for $1 \leq p \leq 2$, $L_p$ has cotype 2, so

$$S_p(l^k_r) \sim d(l^k_r, l^k_2) = k^{1/2 - 1/r}$$

and the constant of equivalence is absolute.

For $2 < p < r$, $L_p$ has cotype $p$ with constant 1 if the average in the definition of cotype is in the $l_p$-sense, so in this range

$$S_p(l^k_r) = d(l^k_r, l^k_p) = k^{1/p - 1/r}.$$

When $2 < p < \infty$, up to a constant depending on $p$, all the normalized 1-symmetric sequences are by Theorem 1.1 of [JMST] just symmetric versions of
Rosenthal’s $X_p$ basis [R], and all $X_p$ spaces are $K_p$-isomorphic to $K_p$-complemented subspaces of $L_p$. Consequently, for any $1 \leq r \leq \infty$, $S_p(l_r^k) \sim \gamma_p(l_r^k)$ up to a constant depending only on $p$. It is possible, but not completely straightforward, to compute $S_p(l_k^r)$ or $\gamma_p(l_k^r)$ up to constants depending on $p$ when $2 < r < p$ by computing the norm of the identity operators between $l_k^r$ and symmetric $X_k^p$ spaces; however, these parameters were first calculated in a different way by Gluskin, Pietsch, and Puhl [GPP, Pi]. They checked that in the range $2 < r < p < \infty$, up to constants depending on $p$, we have

$$\gamma_p(l_r^k) \sim n^\alpha,$$

where $\alpha = (1/r - 1/p)(1/2 - 1/r)/(1/2 - 1/p)$.

**Remark 4.** After the research on this paper was completed, J. Bourgain and L. Tzafriri proved the nice result that condition (iii) in the proof of Proposition 2 implies that the identity operator on $l_r^k$ $K(r)$-factors through $u$ for some $k \geq \varepsilon(r)n$.

**References**


Institute of Mathematics, Polish Academy of Sciences, Gdansk, Poland

Department of Mathematics, Texas A&M University, College Station, Texas

Department of Theoretical Mathematics, Weizmann Institute, Rehovot, Israel

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use