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A BOUNDARY ANALOGUE OF MORERA'S THEOREM IN THE UNIT BALL OF $\mathbf{C}^n$

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ABSTRACT. We show that the boundary values of holomorphic functions in the unit ball of $\mathbf{C}^n$ $(n > 1)$ are characterized by the vanishing of their contour integrals over certain closed curves.

The classical Morera theorem states that if a continuous function f in some region Ω in the complex plane $\mathbf{C}^1$ has zero contour integral around any triangle in Ω then f is holomorphic in Ω. Recently there have appeared some interesting variations of this theorem. The theme of these variations involves changing the class of curves used to test for holomorphy. M. L. Agranovskii and R. É. Val'skii [**2**] have shown that if Ω is the unit ball then it is enough for f to have vanishing nonnegative moments with respect to G-translates of a fixed smooth curve γ, where G is the Moebius group of biholomorphisms of Ω. In detail, the required conditions are

$$\int_{g(\gamma)} z^m f(z)\,dz = 0 \qquad (m = 0, 1, 2, \dots).$$

Here $g(\gamma)$ denotes the image of the curve γ by the biholomorphism g. In [**1**] Agranovskii shows that if f satisfies certain growth conditions, then it is enough to test the zero-moments alone, i.e. it is enough for f to have zero contour integrals over G-translates of γ. Agranovskii also gives a similar result for the case where Ω is a bounded symmetric domain in $\mathbf{C}^n$. In C. Berenstein's paper [**3**] we find, in the case where Ω is the unit ball in $\mathbf{C}^n$ and γ is *not* real analytic, that the same result holds for arbitrary f (i.e. without assuming growth conditions). There are simple examples which show that it is not enough to test for holomorphy using only the family of circles of a fixed radius without assuming a priori growth conditions. On the other hand, L. Zalcman [**7**] has shown that a 'two-radius' family of circles detects holomorphy in certain regions. J. Globevnik [**4**] has formulated conditions for holomorphy using rotation invariant families of curves. In this note we will examine boundary values of holomorphic functions in the unit ball in $\mathbf{C}^n$ $(n > 1)$ and show that they too can be characterized by the vanishing of their contour integrals along curves. We will use two preliminary results.

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PROPOSITION 1 (NAGEL-RUDIN). *Let V be a closed G-invariant subspace of $C(S^{2n-1})$. Then V is one of the following:*

(i) $\{0\}$,
(ii) *the constant functions,*
(iii) *the boundary values of holomorphic functions,*
(iv) *the boundary values of anti-holomorphic functions,*
(v) *the boundary values of pluriharmonic functions,*
(vi) $C(S^{2n-1})$.

By 'boundary values of holomorphic functions' we of course mean continuous functions f on the sphere S^{2n-1} which can be extended to continuous functions F on the unit ball whose restriction to the interior of the ball is holomorphic. Note that (v) is the uniform closure of (iii)+(iv) (see W. Rudin's book [**6**, §13.1.4]).

PROPOSITION 2 (HENKIN-LEITERER). *Let D be a pseudoconvex open set in C^n (not necessarily with smooth boundary) and let Y be a closed complex submanifold of some neighborhood of $\overline{D}$. For every continuous function f on $Y \cap D$ that is holomorphic in $Y \cap D$ there exists a continuous function F on $\overline{D}$ that is holomorphic in D such that $F = f$ on $Y \cap \overline{D}$.*

See Theorem 4.11.1 in Henkin and Leiterer's monograph [**5**]; their theorem is actually more general.

THEOREM. *Let S be a Riemann surface which is a complex submanifold of some open neighborhood of the unit ball $\overline{\mathbf{D}^n} \subset \mathbf{C}^n$ $(n > 1)$, so that $S \cap \partial\overline{\mathbf{D}^n}$ is a smooth curve γ. Denote by z the uniformizing parameter for S. Let f be a continuous function on S^{2n-1} so that for any g a biholomorphism of $\mathbf{D}^n$, we have*

$$\int_{g(\gamma)} f(z)\, dg^* z = 0.$$

Then there exists a function $F \in C(\overline{\mathbf{D}^n})$ which is holomorphic in $\mathbf{D}^n$ and whose restriction to S^{2n-1} is f.

PROOF. Let A be the space of continuous functions satisfying the above condition. Then A is a closed G-invariant subspace of $C(S^{2n-1})$. We now show that there are boundary values of anti-holomorphic functions which do not belong to A. The function $h(z) = \bar{z}$ is a well-defined anti-holomorphic function on S. By the anti-holomorphic version of the extension theorem of Henkin and Leiterer, there is a function $H \in C(\overline{\mathbf{D}^n})$ which extends h and is anti-holomorphic in $\mathbf{D}^n$. Observe that the restriction of H to S^{2n-1} fails to satisfy the above vanishing integral condition. In fact, the uniformizing parameter maps S into a region in $\mathbf{C}^1$ and the corresponding contour integral reduces to an integral of the form $\int \bar{w}\, dw$ over some simple closed curve γ_1 in $\mathbf{C}^1$. By Stokes' theorem, this last integral is just the area enclosed by γ_1 and, in particular, is nonzero. Thus A is a closed G-invariant subspace of $C(S^{2n-1})$ which contains all boundary values of holomorphic functions but does not contain all boundary values of anti-holomorphic functions. By Nagel and Rudin's classification of G-invariant subspaces of $C(S^{2n-1})$, A consists precisely of boundary values of holomorphic functions. Q.E.D.

REMARKS. The converse of the theorem is, of course, trivial. The case where γ is a circle is mentioned in Nagel and Rudin's original paper (see [**6**]). The method

presented here can be generalized to give a corresponding Morera theorem for continuous functions on the Shilov boundary of a bounded symmetric domain in $\mathbf{C}^n$. Also, it is quite likely that the hypothesis of simple-connectivity can be weakened. The proof given uses the uniformizing parameter explicitly so the argument does not extend directly. However, the general method can be applied in any *specific* case where the corresponding contour integral can be directly computed and shown to be nonzero.

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