CYCLIC VECTORS IN $A^{-\infty}$
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ABSTRACT. If $f$ is in $A^{-p}$, then $f$ is cyclic in $A^{-\infty}$ if and only if $f$ is cyclic in every $A^{-q}$ ($q > p$). An analogous result holds for the Bergman spaces $B^p$.

In this note we apply the theory developed in [2 and 3] to explain the relationship between cyclic vectors in $A^{-\infty}$ and $A^{-p}$ or $B^p$.

DEFINITIONS. 1. $A^{-p}$ ($p > 0$) is the Banach space of analytic functions $f(z)$ in $U = \{z \in C \mid |z| < 1\}$ that satisfy $|f(z)| = o[(1 - |z|)^{-p}]$ ($|z| \to 1$) with the norm $||f|| = \max\{|f(z)|(1 - |z|)^p\}$ ($z \in U$). Note that $f_n \to f$ in $A^{-s}$ and $g_n \to g$ in $A^{-t}$ implies $f_n g_n \to fg$ in $A^{-(s+t)}$. Also one can show that if $f_n(z) \not= 0$, $z \in U$, $f(0) = 1$, then $f_n^\alpha \to f^\alpha$ in $A^{-\alpha s}$ ($0 < \alpha < \infty$).

2. $B^p$ ($p > 0$) is the Bergman space, i.e., the "analytic" subspace of $L^p(r \, dr \, d\theta)$ in $U$.

3. $A^{-\infty} = U A^{-p} = U B^p$ ($p > 0$), $A^{-\infty}$ is a linear topological space (see [2, p. 189]); it is the inductive limit of $A^{-p}$.

4. $P$ is the set of all algebraic polynomials $P(z)$. $P$ is dense in any of the spaces $A^{-p}, B^p, A^{-\infty}$.

5. Let $A$ be any of the spaces $A^{-p}, B^p, A^{-\infty}$, and let $f \in A$. The subspace generated by $f$ in $A$ which is invariant under the operator of multiplication by $z$ on $A$ is

$$I(f; A) = \text{clos}\{fP \mid P \in P\} = \text{clos}(fg \mid g \in H^\infty).$$

6. An $f \in A$ is called cyclic in $A$ if $I(f; A) = A$.

THEOREM. If $f \in A^{-p}$ and is cyclic in $A^{-\infty}$ then $f$ is cyclic in every $A^{-q}$ ($q > p$).

A nonvanishing $f$ in $A^{-\infty}$ has a representation in terms of a bounded premeasure $\mu$ (see Proposition 4.1 in [3]). If $f$ is cyclic, $\mu_\sigma$, the $\kappa$-singular part of the premeasure $\mu$ is 0 and the hypothesis of Corollary 3.1.1 in [3] is satisfied. Thus we have the following result.

PROPOSITION. If $f$ is cyclic in $A^{-\infty}$ then there exists a sequence of functions $\{g_m(z)\}$, each belonging to $A^{-\infty}$, such that

(a) $g_m(z) \not= 0$ ($z \in U; m = 1, 2, \ldots$).

(b) $h_m = f g_m$ ($m = 1, 2, \ldots$) belongs to $A^{-s}$ for some fixed $s > 0$.

(c) $\|1 - h_m\|_{A^{-s}} \to 0$ ($m \to \infty$).

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We need the following

**LEMMA.** If \( f \in A^{-p}, \ g \in A^{-\infty}, \ f(z)g(z) \neq 0 \) for \( z \in U \) and \( h = fg \in A^{-s} \) (\( s > p \) then \( h \in I(f; A^{-s}) \).

**PROOF OF THE LEMMA.** Let \( \phi(\alpha) = fg^\alpha = f^{1-\alpha}h^\alpha \in A^{-((1-\alpha)p+\alpha s)} \subseteq A^{-s} \).

Let \( F = \{ \alpha \mid 0 \leq \alpha \leq 1, \ \phi(\alpha) \in I(f, A^{-s}) \} \). \( F \) is closed because \( \phi \) is a continuous function from \([0,1]\) to \( A^{-s} \). Since \( g \in A^{-\infty} \), there is an integer \( n \) so that \( g \in A^{-n} \). If \( \alpha_0 \in F \) and \( \alpha_0 + \varepsilon \in F \) for \( \varepsilon < \frac{1}{n}[(s-(\alpha_0 s + (1-\alpha_0)p)] = \frac{(1-\alpha_0)(s-p)}{n} \).

Since \( g^\varepsilon \in A^{-(1-\alpha_0)(s-p)} \) there exists a sequence of polynomials \( P_n \) such that \( P_n \to g^\varepsilon \) in \( A^{-(1-\alpha_0)(s-p)} \). Since \( P_n fg^{\alpha_0} \in I[f; A^{-s}] \) and \( P_n fg^{\alpha_0} \to fg^{\alpha_0+\varepsilon} = \phi(\alpha_0 + \varepsilon) \) in \( A^{-s} \), we have \( \phi(\alpha_0 + \varepsilon) \in I(f; A^{-s}) \).

Since \( 0 \in F \) we have \( F = [0,1] \) and \( \phi(1) = h \in I(f, A^{-s}) \).

Note that the Proposition and the Lemma imply that \( f \) is cyclic in \( A^{-s} \).

**PROOF OF THE THEOREM.** Let \( \{g_m\}_1^\infty \) be as in the Proposition with \( g_m(0) = 1 \). If \( s \leq p \) then we have \( f \) cyclic in \( A^{-p} \). Let \( s > p \) and assume \( f(0) = 1 \).

Given \( q \) \( p \) we choose an integer \( n \) such that \( s/n < q - p \). By finite induction we show that \( f^{1-k/n}(k = 0, 1, \ldots, n) \) is in \( I[f, A^{-q}] \). If \( k < n \) and \( f^{1-k/n} \in I[f, A^{-q}] \) then

\[
f^{1-k/n}g_m^{1/n} = f^{1-(k+1)/n}h_m^{1/n} \in A^{-(p+s/n)} \subseteq A^{-q}.
\]

By the Lemma, \( f^{1-k/n}_m \in I[f^{1-k/n}, A^{-q}] \subseteq I[f, A^{-q}] \). Since \( h_m^{1/n} \to 1 \) in \( A^{-s/n} \), and \( f^{1-(k+1)/n} \in A^{-p} \),

\[
\lim_{m \to \infty} f^{1-k/n}_m g_m^{1/n} = \lim_{m \to \infty} f^{1-(k+1)/n} h_m^{1/n} f^{1-(k+1)/n} \text{ is in } A^{-q},
\]

and we have \( f^{1-(k+1)/n} \in I(f, A^{-q}) \). Thus we have \( f^0 = 1 \in I(f, A^{-q}) \), i.e., \( f \) is cyclic in \( A^{-q} \).

**COROLLARY TO THE LEMMA.** If \( f \in A^{-p} \) is invertible in \( A^{-\infty} \), i.e., \( |f(z)| \geq c(1-|z|)^\delta \text{ for some positive } \delta \), then \( f \) is cyclic in every \( A^{-q} \) \( q > p \).

We remark that the question whether \( f \) is cyclic (or invertible) in \( A^{-\infty} \) implies \( f \) cyclic in \( A^{-p} \) is still an open question. This question was first posed by H. S. Shapiro (see Theorem 5 in [4] or the remark following Theorem A in [1]).

We also note that a result analogous to our theorem can be proven in a similar manner for the spaces \( B^p \) \( p > 0 \) and that the corresponding corollary to the lemma for the spaces \( B^p \) can be shown to be equivalent to Theorem 5 in [4].

**REFERENCES**


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