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A TAUBERIAN THEOREM FOR HAUSDORFF METHODS

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Dedicated to the memory of Professor V. Ganapathy Iyer

ABSTRACT. Let $H = (H, \chi)$ be a regular Hausdorff summability method defined by the function $\chi \in \mathrm{BV}[0,1]$. It is shown that if χ is absolutely continuous on $[0,1]$, then the methods H and $V \cdot H$ are equivalent for bounded sequences, where V belongs to a certain class of summability methods which includes the Cesàro methods C_α $(\alpha > 0)$, the Abel method A, and the methods $A \cdot C_\alpha$ $(\alpha > -1)$.

1. Introduction. The Hausdorff method $H = (H, \chi)$, where $\chi \in \mathrm{BV}[0,1]$ transforms sequences $s = \{s_k\}$ into sequences $Hs = t = \{t_k\}$ $(k \geqslant 0)$, where

$$t_n = \int_0^1 \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} s_k \, d\chi(t) \qquad (n \geqslant 0). \tag{1}$$

If V_1, V_2 are summability methods, we write $V_1 \sim V_2$ for bounded sequences if V_1 and V_2 are equivalent for bounded sequences, i.e. a bounded sequence is V_1-summable if and only if it is V_2-summable (not necessarily to the same limit). (For detailed properties of the Abel, Cesàro, and Hausdorff methods in general, see [**2** or **8**].) Stam [**7**] showed recently that every C_1-summable bounded sequence is summable by $H = (H, \chi)$ if $\chi(t)$ is absolutely continuous on $[0,1]$. We now prove a stronger result, by making use of a Modified Reduction Principle—a concept introduced by the second author as a valuable tool in Tauberian theory (see e.g. [**3**, **4**, and **5**]).

2. Theorem.

THEOREM. *Let $H = (H, \chi)$ be a Hausdorff method with $\chi(t)$ absolutely continuous on $[0,1]$. Then $V \cdot H \sim H$ for bounded sequences, where $V =$ the Cesàro method C_α $(\alpha > 0)$ or the Abel method A or the method $A \cdot C_\alpha$ $(\alpha > -1)$.*

PROOF. Since V is conservative, it follows that $Hs \in (c)$ implies V sums Hs, i.e. $V \cdot H$ sums s. Now let $s \in (m)$ [the space of bounded sequences] and let $t = Hs$ be defined by (1). We have to prove that

$$s \in (m),\ V \text{ sums } Hs \text{ imply that } Hs \in (c). \tag{2}$$

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Since slow oscillation is a Tauberian condition for V [**6**, Theorem and Remarks 1–3] it is enough to prove that

$$t_n - t_m \to 0 \quad \text{as } n > m \to \infty,\ n/m \to 1. \tag{3}$$

In what follows, we take $n > m$ throughout and limits are always to be as $m \to \infty$, $n/m \to 1$. We write for n, $k \geqslant 0$ and $0 \leqslant t \leqslant 1$:

$$f_{nk} = \binom{n}{k} t^k (1-t)^{n-k}, \qquad F_n(t) = \sum_{k=0}^{n} f_{nk}(t) s_k.$$

Since $\chi(t)$ is absolutely continuous, it is an indefinite integral of a function $g(t)$, say. We have then

$$t_n = \int_0^1 F_n(t) g(t)\, dt. \tag{4}$$

Making an obvious change of variable in the equation obtained by replacing n by m in (4), we have

$$\begin{aligned} t_n - t_m &= \int_0^1 F_n(t) g(t)\, dt - \frac{n}{m} \int_0^{m/n} F_m\left(\frac{n}{m} t\right) g\left(\frac{n}{m} t\right) dt \\ &= \int_0^1 \left\{ F_n(t) g(t) - \frac{n}{m} F_m\left(\frac{n}{m} t\right) g\left(\frac{n}{m} t\right) \right\} dt, \end{aligned}$$

where we take $g(u) = 0$ for $u > 1$. Thus

$$\begin{aligned} t_n - t_m &= \int_0^1 F_n(t) \left\{ g(t) - g\left(\frac{n}{m} t\right) \right\} dt \\ &\quad + \left(1 - \frac{n}{m}\right) \int_0^1 F_n(t) g\left(\frac{n}{m} t\right) dt \\ &\quad + \frac{n}{m} \int_0^1 g\left(\frac{n}{m} t\right) \left[F_n(t) - F_m\left(\frac{n}{m} t\right) \right] dt \\ &= I_1 + I_2 + I_3, \quad \text{say.} \end{aligned}$$

Since $|F_n(t)| \leqslant \sup |s_k| < \infty$

$$I_1 = O\left\{ \int_0^1 \left| g(t) - g\left(\frac{n}{m} t\right) \right| dt \right\} \to 0.$$

Also

$$\int_0^1 F_n(t) g\left(\frac{n}{m} t\right) dt = O(1)$$

and hence $I_2 \to 0$. In order to prove (3) it is therefore enough to prove that $I_3 \to 0$.

Now since the expression inside the square brackets in the integral for I_3 is bounded, the contributions to I_3 of the range $(0, \eta)$ and $(1 - \eta, 1)$ can be made arbitrarily small by choice of η. Thus it is enough to prove that for fixed η with $0 < \eta < \frac{1}{2}$ we have, uniformly in $\eta \leqslant t \leqslant 1 - \eta$,

$$F_n(t) - F_m\left(\frac{n}{m} t\right) \to 0. \tag{5}$$

Since $\{s_k\}$ is bounded, (5) will follow if we show that, uniformly in $\eta \leqslant t \leqslant 1 - \eta$,

$$\sum_{k=m+1}^{n} f_{nk}(t) + \sum_{k=0}^{m} \left| f_{nk}(t) - f_{mk}\left(\frac{n}{m} t\right) \right| \to 0. \tag{6}$$

We now appeal to [2, Theorem 138] (with the "q" of that theorem taken as $(1-t)/t$). Note that while [2] gives the result for fixed t, it does in fact hold uniformly in $\eta \leqslant t \leqslant 1-\eta$.

By clause (4) of that theorem, given $\varepsilon > 0$ there is a number λ such that for sufficiently large n,

$$\sum\nolimits_1 f_{nk}(t) < \varepsilon, \tag{7}$$

where Σ_1 denotes summation over those k for which $|k - nt| > \lambda n^{1/2}$. Since $t \leqslant 1 - \eta$, the range of summation in (7) includes all $k \geqslant n(1-\eta) + \lambda n^{1/2}$, and, since $n/m \to 1$, it will ultimately include the whole range of the first sum in (6). Also, if $|k - nt| > \lambda n^{1/2}$, then

$$\left|k - m \cdot \frac{n}{m} t\right| > \lambda n^{1/2} > \lambda m^{1/2},$$

so that the following inequality similar to (7) holds also:

$$\sum\nolimits_1 f_{mk}\left(\frac{n}{m} t\right) < \varepsilon.$$

Thus we need only show that for fixed λ,

$$\sum{}^{*}\left|f_{nk}(t) - f_{mk}\left(\frac{n}{m} t\right)\right| \to 0 \tag{8}$$

(where Σ^* denotes summation over those k for which $|k - nt| \leqslant \lambda n^{1/2}$), uniformly in $[\eta, 1-\eta]$. In this range of values for k, clause (5) of [2, Theorem 138] is applicable, so that, uniformly in the relevant range of values for k,

$$f_{nk}(t) = [2\pi n t(1-t)]^{-1/2} A(n,k,t)[1 + O(n^{-1/2})], \tag{9}$$

where

$$A(n,k,t) = \exp\left\{\frac{-(k-nt)^2}{2nt(1-t)}\right\}.$$

Also, replacing n by m and t by $(n/m)t$, we get

$$f_{mk}\left(\frac{n}{m} t\right) = \left[2\pi n t\left(1 - \frac{n}{m} t\right)\right]^{-1/2} A\left(m, k, \frac{n}{m} t\right)[1 + O(n^{-1/2})]. \tag{10}$$

Since

$$\sum [2\pi n t(1-t)]^{-1/2} A(n,k,t) = O(1), \tag{11}$$

the contribution to (8) of the "O"-term in (9) is $O(n^{-1/2})$. Similarly the contribution to (8) of the "O"-term in (10) is $O(n^{-1/2})$. Thus, omitting the factor $(2\pi t)^{-1/2}$ (as we may, since this is bounded), it is enough to prove that

$$n^{-1/2} \sum{}^{*}\left|(1-t)^{-1/2} A(n,k,t) - \left(1 - \frac{n}{m} t\right)^{-1/2} A\left(m, k, \frac{t}{m}\right)\right| \to 0, \tag{12}$$

where the symbol Σ^* denotes summation over the values of k for which $|k - nt| \leqslant \lambda n^{1/2}$. The expression on the left-hand side of (12) is less than or equal to

$$\begin{aligned} & n^{-1/2}\sum{}^*\left|(1-t)^{-1/2} - \left(1 - \frac{n}{m}t\right)^{-1/2}\right| A(n,k,t) \\ & \quad + n^{-1/2}\sum{}^*\left(1 - \frac{n}{m}t\right)^{-1/2}\left|A(n,k,t) - A\left(m,k,\frac{n}{m}t\right)\right| \\ & = J_1 + J_2, \quad \text{say.} \end{aligned}$$

Thus, in order to prove (8) and the Theorem, it is enough to prove that J_1 and J_2 tend to 0 uniformly. Now since the function $h(t) = (1-t)^{-1/2}$ has a bounded derivative in the interval $\eta \leqslant t \leqslant 1 - \eta$, the expression inside the modulus in J_1 is

$$h(t) - h\left(\frac{n}{m}t\right) = O\left(\left|1 - \frac{n}{m}\right|t\right) = o(1).$$

It follows from (11) that $J_1 \to 0$.

Now, if $0 < p < q$, then

$$0 < e^{-p} - e^{-q} = \int_p^q e^{-x}\,dx < (q-p)e^{-p}.$$

Hence the expression inside the modulus in J_2 is positive and less than

$$\begin{aligned} & (k-nt)^2\left[\frac{1}{2nt(1-(n/m)t)} - \frac{1}{2nt(1-t)}\right]A(n,k,t) \\ & \quad = (k-nt)^2\frac{(n/m)-1}{2n(1-(n/m)t)(1-t)}A(n,k,t). \end{aligned}$$

Since $n/m - 1 = o(1)$ and the function $(1-t)^{-1}$ and $(1-(n/m)t)^{-1}$ are bounded, it follows that

$$J_2 = o\left\{n^{-3/2}\sum{}^*(k-nt)^2A(n,k,t)\right\}.$$

But the expression in curly brackets is bounded. Hence $J_2 \to 0$, and the proof of the Theorem is complete.

REMARKS. (1) The Theorem gives a sufficient condition in order that a conservative Hausdorff method $H = (H, \chi)$ will be equivalent to $V \cdot H$ for bounded sequences. But the absolute continuity of χ is not *necessary* for this equivalence, as is shown by the following example: For an arbitrary number λ, let $K_\lambda = (H, g)$ be the Hausdorff matrix generated by the function g with $g(0) = 0$ and $g(t) = \lambda$ for $0 < t \leqslant 1$. Then K_λ sums every bounded sequence $\{s_n\}$ to λs_0, and hence for any Hausdorff method $H = (H, \chi)$ with $\chi \in \mathrm{AC}[0,1]$, $H + K_\lambda \sim C_1(H + K_\lambda)$. But the generating function $\chi + g$ of $H + K_\lambda \notin \mathrm{AC}[0,1]$ if $\lambda \neq 0$.

(2) Subsequent to the completion of this paper we have succeeded in proving that a conservative Hausdorff method P will be equivalent to $V \cdot P$ for bounded sequences if and only if P is of the form $H + K_\lambda$, where $H = (H, \chi)$ with $\chi \in \mathrm{AC}[0,1]$. In particular, if $H = (H, \chi)$ is a multiplicative Hausdorff method, then $H \sim V \cdot H$ for bounded sequences if and only if $\chi \in \mathrm{AC}[0,1]$. Thus the Theorem is a best possible one for multiplicative Hausdorff methods in the sense that the condition $\chi \in \mathrm{AC}[0,1]$ cannot be weakened.

(3) The Theorem is also best possible in the sense that it does not hold for sequences bounded on only one side. To see this, it is enough to consider the case $V = C_\alpha$ $(0 < \alpha < 1)$, since of $0 < \alpha < \beta$, $C_\alpha H \subset C_\beta H \subset A \cdot H$. Take $H = C_{1-\alpha}$ and a sequence s such that $s_n \geqslant 0$, $C_1 s \in (c)$ [and hence $C_\alpha \cdot C_{1-\alpha} s \in (c)$] but $C_{1-\alpha} s \notin (c)$ (e.g. let $s_n = 2^{(1-\alpha)k}$ if $n = 2^k$ and $s_n = 0$ otherwise, for $n, k = 0, 1, \ldots$).

(4) The proof of the Theorem shows that the result is true for an arbitrary conservative method V for which slow oscillation is a Tauberian condition. It is therefore true for any conservative method V which has $na_n = O(1)$ as a Tauberian condition and for which $V \subset V \cdot C_1$. (To see this, adapt the theorem and proof given in [**6**].) The methods V considered in the Theorem of the present paper are all of this type.

(5) If A, B are matrices, let us write $A \approx B$ if $(A - B)x \in (c)$ for every $x \in (m)$ (see [**8**, p. 69] on "volläquivalenz"). Our Theorem says that if $\chi \in \mathrm{AC}[0,1]$ and $H = (H, \chi)$, then $H \sim HC_1$ for bounded sequences; however we cannot assert that

$$H \approx HC_1 \quad \text{i.e.} \quad (H - HC_1)x \in (c) \text{ for every } x \in (m). \tag{13}$$

For suppose that (13) holds when $H = C_1$. (C_1 has generating function $\chi(t) = t \in \mathrm{AC}[0,1]$.) Since $C_1 - C_1C_1$ is multiplicative, it follows (by [**2**, Theorem 3]) that, for every $x \in (m)$,

$$C_1(I - C_1)x = (C_1 - C_1C_1)x \in (c_0) \tag{14}$$

and hence

$$C_2(I - C_1)x = (C_2 - C_2C_1)x \in (c_0). \tag{15}$$

We have then both the relations

$$C_1C_1 \approx C_1 \quad \textit{and} \quad C_1C_2 \approx C_2. \tag{16}$$

Since $C_1C_1 - C_2 = (C_1C_2 - C_2)/2$ (see [**2**, p. 107 or **8**, p. 108]), it follows from (16) that $C_1C_1 \approx C_2$ and hence $C_1 \approx C_1C_2 \approx C_2$. Since $C_1 - C_2$ is multiplicative and $C_1 \approx C_2$, we must have then

$$(C_1 - C_2)x \in (c_0) \quad \text{for every bounded sequence } x. \tag{17}$$

But Cooke [**1**, pp. 118–119] has proved that (17) is not true. Hence (13) is false when $H = C_1$.

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