THE GELFAND THEOREM AND ITS CONVERSE
FOR KÄHLER MANIFOLDS

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ABSTRACT. We characterize the locally Hermitian symmetric manifolds
among the homogeneous Kähler manifolds $M$ by each of the following
properties:

(i) all $A_0(M)$-invariant differential operators on $M$ commute ($A_0(M)$
denotes the identity component of the group of all holomorphic isometries);
(ii) all geodesics are orbits of one-parameter groups of holomorphic isome-
tries.

Recently, D’Atri, Dorfmeister, and Zhao Yan da [1] proved the following character-
ization of symmetric Siegel domains.

THEOREM A. Let $D$ be a homogeneous Siegel domain and $G$ the identity com-
ponent of the automorphism group of $D$. Then the algebra of $G$-invariant
differential operators on $D$ is commutative if and only if $D$ is a symmetric domain.

The “if” part is an easy modification of the well-known Gelfand theorem (see [3]). It still holds in the following more general situation. Let $M$ be a Hermitian
symmetric space (thus a homogeneous Kähler manifold), and let $A_0(M)$ denote
the identity component of the group of all holomorphic isometries of $M$. Then the
algebra of $A_0(M)$-invariant differential operators on $M$ is commutative.

In this report we prove a converse of the last statement and this gives an essential
generalization of Theorem A.

THEOREM 1. Let $(M,g,J)$ be a homogeneous Kähler manifold. If all $A_0(M)$-
invariant differential operators on $M$ commute, then $M$ is locally Hermitian sym-
metric.

COROLLARY. Let $M$ be a simply connected homogeneous Kähler manifold. Then
the algebra of $A_0(M)$-invariant differential operators on $M$ is commutative if and
only if $M$ is Hermitian symmetric.

The proof is based on two lemmas.

LEMMA 1 [7]. Let $(M,g)$ be a smooth $n$-dimensional Riemannian manifold. Let

$$D = \sum_{i_1, \ldots, i_k = 1}^n T^{i_1 \ldots i_k} \nabla_{i_1 \ldots i_k}$$

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be a differential operator whose coefficients are symmetric and which commutes with
the Laplacian \( \Delta = \sum_{j=1}^{n} g^{ij} \nabla_{ij}^2 \). Then the following tensor identity holds:

\[
\nabla_{i_1} T_{i_2 \ldots i_{k+1}} + \cdots + \nabla_{i_k} T_{i_1 \ldots i_k} = 0.
\]

A short proof can be found in [4].

**Lemma 2.** A Kähler manifold \((M, g, J)\) is locally Hermitian symmetric if and only if

\[
(\nabla_X R)(X, JX, X, JX) = 0
\]

holds for each tangent vector \(X\).

The identity (2) was first investigated by A. Gray in connection with the theory of 3-symmetric spaces, and the proof of Lemma 2 can be obtained by combining Theorem 4.6 and Corollary 4.4 from [2]. A full direct proof (elementary, but rather long) can be found in two parts in [6] and [8]. Although this lemma proved to be useful in many topics of Kählerian geometry, a really short and simple proof is not yet available.

**Proof of Theorem 1.** Define a 4-valent tensor field \(T\) on \(M\) by putting

\[
T_{ijkl} = T_{ijkl} = R(e_i, J e_j, e_k, J e_l) + R(e_k, J e_j, e_i, J e_l) + R(e_i, J e_j, e_k, J e_l)
\]

with respect to any orthonormal frame (where \(R\) is the curvature tensor). Since \(M\) is Kählerian, \(T_{ijkl}\) is symmetric. Further, \(T\) is \(A_0(M)\)-invariant and hence the differential operator

\[
D = \sum_{i,j,k,l=1}^{n} T_{ijkl} \nabla_{ijkl}^4
\]

is also \(A_0(M)\)-invariant. Then \(D\) must commute with \(\Delta\) and Lemma 1 gives

\[
(\nabla_X D)(X, X, X, X) = 0,
\]

i.e. \((\nabla_X R)(X, JX, X, JX) = 0\) for each tangent vector \(X\). Now we use only Lemma 2 to obtain the result.

**Note A.** The converse of the Gelfand theorem is not true in the real case. For instance, there are many naturally reductive homogeneous Riemannian manifolds for which all \(I_0(M)\)-invariant differential operators commute and which are not locally symmetric. Yet, the following weaker converse still holds (see [4]): Let \((M, g)\) be a homogeneous Riemannian manifold and \(I_0(M)\) the identity component of the full isometry group of \(M\). If all \(I_0(M)\)-invariant differential operators on \(M\) commute, then the local geodesic symmetries of \((M, g)\) are volume-preserving.

**Note B.** The following result is related to our Theorem 1.

**Theorem 2.** Let \((M, g, J)\) be a homogeneous Kähler manifold all of whose geodesics are orbits of one-parameter groups of holomorphic isometries. Then \((M, g, J)\) is locally Hermitian symmetric.

**Proof.** Along any fixed geodesic \(\gamma\), the function \(R(\gamma', J \gamma', \gamma', J \gamma')\) is constant. Hence \(\gamma'[R(\gamma', J \gamma', \gamma', J \gamma')] = 0\), and because \(M\) is Kählerian, we get the condition of Lemma 2.

Again, a real analogue holds in the weaker form (see [5]): Let \((M, g)\) be a homogeneous Riemannian manifold all of whose geodesics are orbits of one-parameter groups of isometries. Then the local geodesic symmetries of \((M, g)\) are volume-preserving.
REFERENCES


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