ON IMMERSED COMPACT SUBMANIFOLDS OF EUCLIDEAN SPACE

MARCO RIGOLI

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ABSTRACT. Given an immersion \( f: M \to \mathbb{R}^n \) of a compact Riemannian manifold \( M \) we prove a simple criterion involving the tension field of \( f \) to determine whether or not \( f \) is an isometry.

1. Introduction. Let \( f: M \to \mathbb{R}^n \) be an immersion of a Riemannian manifold into Euclidean space. A natural problem is to determine whether or not \( f \) is an isometry. In this note we give a proof of the following simple result (see §2 for details).

THEOREM. Let \( M \) be an \( m \)-dimensional, compact, oriented, Riemannian manifold with metric \( ds^2 \) and let \( f: M \to \mathbb{R}^n \) be an immersion. Set \( ds^2 \) for the induced metric on \( M \) via \( f \), \( u \) for the ratio of the volume elements, \( \tau \) for the tension field of \( f \) and \( H \) for the mean curvature vector of \( f: (M, ds^2) \to \mathbb{R}^n \). Then \( f \) is an isometry iff

(i) \( \langle f, \tau - umH \rangle \geq 0 \) and

(ii) \( f \) is volume decreasing for \( m > 3 \),

(iii) \( f \) is volume preserving for \( m = 2 \),

(iv) \( f \) is volume increasing for \( m = 1 \).

REMARKS. 1. The necessity of the above conditions is clear. Indeed if \( f \) is an isometry then \( u = 1 \), that is \( f \) is volume preserving, and \( \tau = mH \) (see §2).

2. For \( m = 2 \) in the proof of the theorem it will become apparent that (i) alone implies that \( f \) is conformal. We wish to state this in the form of the following:

PROPOSITION. Let \( f: M \to \mathbb{R}^n \) be an immersed compact Riemannian surface. Then \( f \) is conformal iff \( \tau = 2uH \).

PROOF. Sufficiency follows from above. Necessity follows from a well-known computation of the tension field (for instance see Hoffman-Osserman [1]).

3. A step in the proof of the theorem is based on the following result from linear algebra. Let \( V \) be a real \( m \)-dimensional vector space, \( G \) an inner product in \( V \) and \( H \) a symmetric semi-positive-definite bilinear form. Let \( (g_{ij}), (h_{ij}) \) be their matrices with respect to a basis of \( V \). Set \( g = \det(g_{ij}) \) and \( h = \det(h_{ij}) \); clearly \( g > 0 \) and \( h \geq 0 \). For \( \lambda \) a parameter consider the determinant

\[
\det(g_{ij} + \lambda h_{ij}) = g + mP\lambda + \cdots + h\lambda^m,
\]

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where $P$ is a polynomial in the entries of the matrices of $G$ and $H$. It is easily verified that the quantity $P/g$ is independent of the basis chosen in $V$; we claim that

$$P/g \geq (h/g)^{1/m}$$

where the equality sign holds iff $h_{ij} = \rho g_{ij}$ for a certain $\rho$. Indeed we choose a basis of $V$ such that $g_{ij} = \delta_{ij}$ and $(h_{ij})$ is diagonal so that $(h_{ij}) = \text{diag}(\lambda_1, \ldots, \lambda_m)$. Then (1) becomes

$$\frac{1}{m} \sum_{i=1}^{m} \lambda_i \geq \left( \prod_{i=1}^{m} \lambda_i \right)^{1/m}$$

and the result is known as a standard inequality.

2. Preliminaries on differential geometry. We realize the Euclidean space $\mathbb{R}^n$ as the homogeneous space $E(n)/SO(n)$, where $E(n) = SO(n) \times \mathbb{R}^n$ is the group of rigid motions and $SO(n)$ its isotropy subgroup at the origin $0$ of $\mathbb{R}^n$. From now on we fix the indices convention $1 \leq A, B, \ldots \leq n$, $1 \leq i, j, \ldots \leq m$, $m + 1 \leq \alpha, \beta, \ldots \leq n$. If $\Theta^B_A, \Theta^A$ denote the components of the Maurer-Cartan form of $E(n)$ and $s$ is a local section of the bundle $E(n) \to \mathbb{R}^n$ the forms

$$\theta^A = s^* \Theta^A$$

give a local orthonormal coframe in $\mathbb{R}^n$ whose corresponding Levi-Civita connection forms are

$$\theta^A_B = s^* \Theta^A_B.$$

From now on we will drop the pull-back notation because it will be clear from the context where the forms must be considered. Let $f: M \to \mathbb{R}^n$ be an immersion of an $m$-dimensional manifold. A Darboux frame along $f$ is a (locally defined) smooth function $e$ on $M$ with values in $E(n)$ of the form

$$e: p \to (e_A(p), f(p))$$

where $e_A(p)$ are the columns of an $SO(n)$ matrix such that the vectors $e_i(p)$ span the image of the tangent space of $M$ at $p$ under the differential of $f$ and determine the correct orientation. It follows that on $M$

$$de_A = \theta^B_A \otimes e_B,$$

$$\theta^\alpha = 0.$$

In particular (6) implies that the metric $d\sigma^2$ induced by $f$ on $M$ can be written as

$$d\sigma^2 = \sum_i (\theta^i)^2.$$

Suppose now $M$ is an oriented Riemannian manifold with metric $ds^2$. Let $\phi^i$ be an oriented orthonormal (local) coframe on it with corresponding connection forms $\theta^i_j$. On the common domain of definition of the $\theta^A$ and $\phi^i$'s we have

$$e^A = B^A_j \phi^j.$$
for some smooth function $B^\alpha_j$. According to (6)
\begin{equation}
B^\alpha_j = 0.
\end{equation}
In particular the volume element $d\tilde{V}$ of the metric $d\sigma^2$ can be expressed as
\begin{equation}
d\tilde{V} = \det(B^\alpha_j)dV
\end{equation}
where $dV$ is the volume element of the metric $ds^2$; equivalently their ratio is given by the positive function
\begin{equation}
u = \det(B^\alpha_j).
\end{equation}
The immersion $f$ will be said to be volume decreasing if at every point $p \in M$
\begin{equation}
u \leq 1.
\end{equation}

Volume increasing and volume preserving are defined analogously. Exterior differentiation of (6) and (8) and use of the structure equations of $\mathbb{R}^n$ and $(M, ds^2)$ gives:
\begin{equation}
dB^A_i - B^A_j \phi^j_i + B^B_i \phi^A_B = B^A_{ij} \phi^j
\end{equation}
for some smooth functions $B^A_{ij}$ such that $B^A_{ij} = B^A_{ji}$. The $B^A_{ij}$'s are the coefficients of the (generalized) second fundamental tensor of the immersion $f: (M, ds^2) \to \mathbb{R}^n$, i.e.
\begin{equation}
\nabla df = B^A_{ij} \phi^j \otimes \phi^i \otimes e_A
\end{equation}
whose trace with respect to $ds^2$ gives the tension field $\tau$ of $f$, i.e.
\begin{equation}
\tau = B^A_{ii} e_A.
\end{equation}

We remark that if instead of considering $f: (M, ds^2) \to \mathbb{R}^n$ we consider $f: (M, d\sigma^2) \to \mathbb{R}^n$ the above procedure gives the second fundamental tensor and $m$ times the mean curvature vector $H$.

We denote by $\Delta_{ds^2}$, $\Delta_{d\sigma^2}$ the Laplace-Beltrami operators relative to $ds^2$ and $d\sigma^2$. We now claim
\begin{equation}
\frac{1}{2} \Delta_{ds^2} |f|^2 = \langle f, \tau \rangle + \|df\|^2
\end{equation}
and similarly
\begin{equation}
\frac{1}{2} \Delta_{d\sigma^2} |f|^2 = m \{ \langle f, H \rangle + 1 \}.
\end{equation}
In the above formulas $\langle \ , \ \rangle$ is the usual inner product in $\mathbb{R}^n$ and $||$ its corresponding norm, while $\| \|$ is the Hilbert-Schmidt norm of $df$; that is
\begin{equation}
\|df\|^2 = \sum_{i,A} (B^A_i)^2.
\end{equation}
The proof of (16) is a standard computation. Indeed by (6) and (8) we have
\begin{equation}
d|f|^2 = 2B^A_i \langle f, e_A \rangle \phi^i
\end{equation}
and by (5), (6), (8), (13)
\begin{equation}
d(2B^A_i \langle f, e_A \rangle) - 2B^A_j \langle f, e_A \rangle \phi^j_i = 2\langle f, e_A \rangle B^A_{ij} + B^A_i B^A_j \phi^j.
\end{equation}
By definition $\Delta_{ds^2}|f|^2$ is the trace of the coefficients appearing in the right-hand side of (19), hence by (15) and (18) we obtain (16).

In case $M$ is compact, integration of (16) gives

$$E(f) = -\frac{1}{2} \int_M \langle f, \tau \rangle \, dV$$

where $E(f)$ is the energy of $f$. If $f$ is an isometry (20) generalizes a formula of Minkowski on convex bodies.

3. Proof of the theorem. We just prove sufficiency. Since $M$ is compact, integrating (16), (17) and using (10), (11) we obtain

$$\int_M \{\langle f, \tau - umH \rangle + \|df\|^2 - um\} \, dV = 0.$$  

We now let $ds^2$ and $d\sigma^2$ play the role of $G$ and $H$ in the introduction. Our considerations will be pointwise. The matrix of $ds^2$ with respect to the basis $\phi^i$ is of course the identity ($\delta_{ij}$), while from (7) and (8) we get

$$d\sigma^2 = B^k_i B^k_j \phi^i \phi^j$$

showing that the matrix of $d\sigma^2$ with respect to the same basis is $(B^k_i B^k_j)$. In particular from (11) its determinant is $u^2$. A simple computation shows that in this case $P = \frac{1}{m} \|df\|^2$. From (1) we therefore obtain

$$\|df\|^2 \geq mu^{2/m},$$

and hence

$$\|df\|^2 - um \geq m(u^{2/m} - u).$$

On the other hand by (i) and (21) we get $\int(\|df\|^2 - um) \leq 0$. Thus, if

$$u^{2/m} - u \geq 0,$$

combining with (23) gives

$$u = u^{2/m}.$$  

We deduce that equality holds in (22), hence

$$B^k_i B^k_j = \rho \delta_{ij};$$

that is, the map $f$ is conformal. Now in case $m \geq 3$ (24) follows from (j); moreover from (25) we deduce $u = 1$ which implies $\rho = 1$ in (26), i.e. $f$ is an isometry. The remaining two cases are handled similarly.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DURHAM, DURHAM, ENGLAND

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS, 34100 TRIESTE, ITALY