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# A NOTE ON HYPERHERMITIAN FOUR-MANIFOLDS

CHARLES P. BOYER

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ABSTRACT. It is shown that the only hyperhermitian four-manifolds are, up to conformal equivalence, tori and $K3$ surfaces with their standard hyper-Kähler structures and certain conformally flat Hopf surfaces.

**Introduction.** Hypercomplex manifolds are, roughly speaking, complex manifolds with a two-sphere's worth of complex structures. They are generalizations of both hyper-Kähler manifolds [**5**] and quaternionic manifolds in the sense of Sommese [**15**]. (This is a much stronger notion than the more widely accepted quaternionic manifold or quaternionic Kähler manifold used by others; cf. [**14**, **18**].) It is easy to see from the definition and the Enriques-Kodaira classification of compact complex surfaces that the only compact hyper-Kähler manifolds in real dimension four are complex tori and $K3$ surfaces. Moreover, Kato [**10**] showed that the only compact quaternionic manifolds of dimension four in the sense of Sommese are tori and Hopf surfaces. In this note it is shown that these three types of surfaces exhaust the compact hyperhermitian four-manifolds in dimension four up to conformal equivalence. Our theorem can then be used to simplify the proof of Kato's theorem [**10**], avoiding the case-by-case analysis that he makes. We also generalize (in the case of dimension four) a result of Sommese [**15**] which states that the twistor space of a hyperhermitian four-manifold $M$ (not necessarily compact) fibers holomorphically over the Riemann sphere $\mathbf{P}^1$.

A smooth manifold $M$ is called *almost hypercomplex* if there are two almost complex structures $I_1$ and $I_2$ on $M$ satisfying $I_1I_2 + I_2I_1 = 0$. When this is the case, we can define another almost complex structure $I_3$ by $I_3 = I_1I_2$. These three almost complex structures on $M$ then satisfy the algebra of the quaternions, namely,

$$I_iI_j + I_jI_i = 0, \quad i \neq j; \qquad I_i^2 = -1, \quad i = 1, 2, 3.$$

This is a Lie superalgebra $\mathfrak{g} \simeq \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with generators $1 \in \mathfrak{g}_0$ and $I_i \in \mathfrak{g}_1$. Moreover, there is an embedding $S^2 \to \mathfrak{g}_1$ giving a two-sphere's worth of almost complex structures on $M$ given by

$$I_x = \sum_{i=1}^{3} x_i I_i$$

so that $I_x^2 = -1$ iff $x_1^2 + x_2^2 + x_3^2 = 1$. We shall frequently view the two-sphere as the complex projective line $\mathbf{P}^1$ and use homogeneous coordinates $z = (z_1, z_2)$

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to parametrize the almost complex structure, writing $I_z$. If the almost complex structure $I_z$ is integrable for all $z \in \mathbf{P}^1$, $M$ is called *hypercomplex* [**6**]. $M$ is called *quaternionic in the sense of Sommese* [**15**] or *coordinate quaternionic* if the transition functions $\phi_j \circ \phi_i^{-1}$: $\phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ are quaternionic maps, i.e. satisfy $f^* \circ I_z = I_z \circ f_*$ for all $z \in \mathbf{P}^1$. Here $\{(U_i, \phi_i)\}$ denotes an atlas of charts on $M$. Sommese [**15**] has shown that quaternionic maps are necessarily restrictions of affine maps. It follows that a *coordinate quaternionic manifold is hypercomplex.* The converse is *not* true; a $K3$ surface is hypercomplex but not affine.

Now let $g$ be a Riemannian metric on $M$. If $M$ is (almost) hypercomplex and

$$g(I_z X, I_z Y) = g(X, Y)$$

for all vector fields $X, Y$ on $M$ and all $z \in \mathbf{P}^1$, then the triple $(M, g, I_z)$ is called an (*almost*) *hyperhermitian manifold.* Associated with every (almost) hyperhermitian manifold there is a $\mathbf{P}^1$'s worth of two-forms on $M$, namely,

$$\omega_z(X, Y) = g(X, I_z Y).$$

We shall refer to $\omega_z$ as the *hyperhermitian 2-form* associated to $(M, g, I_z)$. If this two-form is closed for all $z \in \mathbf{P}^1$, $(M, g, I_z)$ is called *hyper-Kähler.*

It is the purpose of this note to prove the following

THEOREM 1. *Let $(M, g, I_z)$ be a compact hyperhermitian four-manifold. Then $(M, g)$ is conformally equivalent to one of the following:*

(1) *A torus with its flat metric.*

(2) *A $K3$ surface with a hyper-Kähler Yau metric.*

(3) *A coordinate quaternionic Hopf surface with its standard locally conformally flat metric.*

REMARKS. The general Hopf surfaces have been studied in detail by Kodaira [**12**] and Kato [**11**]. They are all $S^3/H$ bundles over $S^1$ where $H$ is a certain finite group. Kato [**10**] has given a list of all Hopf surfaces that admit a coordinate quaternionic structure [**10**, Proposition 8]. In the latter case, the surface admits the standard conformally flat structure.

We also give a result generalizing (in dimension four) a result of Sommese [**15**], namely, if $M$ is hyperhermitian then there is a holomorphic fibration of the twistor space $\mathbf{P}V_+M$ over the Riemann sphere $\mathbf{P}^1$. The twistor space for a complex torus was described in [**3**] while that for certain primary Hopf surfaces was described in [**9**].

This note was inspired by the beautiful lectures of Nigel Hitchin given at the Seminaire de Mathematique Supérieures at the Université de Montréal, July 29–August 16, 1985, where the author became aware of the rich structure of hyper-Kähler manifolds. I would also like to thank Eugenio Calabi, Nigel Hitchin, and Ben Mann for stimulating discussions.

**1. Preliminaries.** There are several important features of Hermitian geometry in complex dimension two which give it a rather rich structure. First, in real dimension four, a special role is played by the Hodge star operator*. It is an involution on the bundle of two-forms $\Lambda^2 M$ and this splits $\Lambda^2 M$ into eigenspaces, viz.,

$$\Lambda^2 M \simeq \Lambda^2_+ M \oplus \Lambda^2_- M. \tag{1.1}$$

But there is another splitting of two-forms due to the complex structure, namely,

$$\Lambda^2 M \otimes \mathbb{C} \simeq \Lambda^{2,0} M \oplus \Lambda^{1,1} M \oplus \Lambda^{0,2} M \tag{1.2}$$

and the two splittings are related by

$$\Lambda^2_- M \otimes \mathbb{C} \subset \Lambda^{1,1} M, \qquad \Lambda^2_+ M \otimes \mathbb{C} \simeq \Lambda^{2,0} M \oplus \Lambda^{0,2} M \oplus \mathbb{C}, \tag{1.3}$$

where $\mathbb{C}$ is the trivial complex line bundle on $M$ generated by the fundamental hermitian two-form $\Omega$. The second special feature of complex dimension two is that the map $L: \Lambda^1 M \to \Lambda^3 M$, defined by $L\alpha = \alpha \wedge \Omega$, is an isomorphism. Thus, associated with every hermitian metric $g$ on $M$ there is a natural one-form $\beta$ defined by

$$d\Omega + \beta \wedge \Omega = 0. \tag{1.4}$$

So $\beta$ vanishes identically if and only if $g$ is Kähler, and thus provides a convenient way to study the non-Kähler case. Finally, Riemannian geometry in four dimensions has a convenient spin representation in terms of quaternions arising from the well-known exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \to \mathrm{SO}(4) \to 1. \tag{1.5}$$

For general $M$ the complexified spin bundles $V_\pm M$ are defined only locally, but the projective spin bundles $\mathbf{P}V_\pm M$ and the symmetric tensor product bundles $S^2 V_\pm M$ are globally defined. Furthermore, there is an important isomorphism

$$\Lambda^2_\pm M \otimes \mathbb{C} \simeq S^2 V^*_\pm M. \tag{1.6}$$

To describe explicitly the spin representation of the orthonormal frame bundle, we fix a frame $\{X_a : a = 0, \dots, 3\}$ or equivalently a coframe $\{\theta^a : a = 0, \dots, 3\}$ and construct the matrix

$$H = \begin{pmatrix} \theta^0 + i\theta^3 & \theta^1 + i\theta^2 \\ -\theta^1 + i\theta^2 & \theta^0 - i\theta^3 \end{pmatrix} \tag{1.7}$$

and view this as an element of the tensor product $V^*_+ \otimes V^*_- \simeq T^*_x M \otimes \mathbb{C}$. Since the metric at $x \in M$ is just $\det H$, the group $\mathrm{SO}(4)$ acting on the coframe $\{\theta^a\}$ is represented by right and left matrix multiplication by the group $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$. Now on the complex spinor spaces $V^*_\pm$, the group $\mathrm{SU}(2) \otimes \mathbb{C} \simeq \mathrm{SL}(2, \mathbb{C})$ leaves invariant a complex two-form and thus gives $V^*_\pm$ natural complex symplectic structures $\varepsilon$ represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. With the identification $T_x M \otimes \mathbb{C} \simeq V_+ \otimes V_-$, the metric $g$ is identified with the pairing on $V_+ \otimes V_- \times V_+ \otimes V_-$ induced by $\varepsilon \otimes \varepsilon$, so $\varepsilon$ can be thought of as kind of "square root" of the metric $g$. Furthermore, the real structure on $V_+ \otimes V_-$ defined by complex conjugation is given by the composition of the natural isomorphisms $V_+ \otimes V_- \simeq V^*_+ \otimes V^*_- \simeq V_+ \otimes V_-$ induced by $\varepsilon \otimes \varepsilon$ followed by that induced by the metric.

This spin representation has a well-known convenient description [**7**] in terms of quaternions. We define the quaternionic-valued one-form at $x \in M$ by $\theta = \theta^0 + i\theta^3 + j\theta^1 + k\theta^2$, where $1, i, j, k$ are the standard basis for the quaternions $\mathbf{H}$. Thus we have $V^*_+ \otimes V^*_- \simeq T^*_x M \simeq \mathbf{H}$. Letting $M(2, \mathbb{C})$ denote the $2 \times 2$ matrices over $\mathbb{C}$, this isomorphism is described explicitly by the map $A: \mathbf{H} \to M(2, \mathbb{C})$ defined by $A \circ \theta = H$. On $\mathbf{H}$, the group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ becomes right and left multiplication by

the group Sp(1) of invertible quaternions. The Riemannian metric can be realized locally as

$$g = \sum_{a=0}^{3} \theta^a \theta^a = \det H = |\theta|^2, \tag{1.8}$$

where $|\theta|^2$ is the quaternionic norm. Suppose now we try to realize $TM$ as a quaternionic line bundle on $M$ and the metric $g$ as a quaternionic norm in this line bundle. In order to do so we need bundle maps $I_i: TM \to TM$ satisfying $I_i^2 = -1$, $I_i I_j = I_k$ cyclicly, and $I_i I_j + I_j I_i = 0$, that is, $(M, g)$ is an almost hyperhermitian manifold. Equivalently, we can give a quaternionic line bundle in terms of its transition functions $\psi_{\alpha\beta}$: $U_\alpha \cap U_\beta \to \mathrm{Sp}(1)$ where $\{U_\alpha\}$ is an open cover of $M$. We have arrived at:

PROPOSITION 1. *A Riemannian four-manifold $(M, g)$ is almost hyperhermitian if and only if there is a reduction of the frame bundle $F(M)$ with group* $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-/\mathbb{Z}_2 \simeq \mathrm{Sp}(1) \times \mathrm{Sp}(1)/\mathbb{Z}_2$ *to the group* $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)_-$.

An immediate corollary of this is that an almost hyperhermitian four-manifold is a spin manifold. In [**4**] the author classified those compact complex surfaces which are spin manifolds and admit an anti-self-dual metric. Since, as we shall show, hyperhermitian four-manifolds are anti-self-dual, we could make use of this classification; however, we prefer to give a direct proof which is similar to that in [**4**].

**2. Hyperhermitian four-manifolds.** Let $\Gamma = \Gamma_+ \oplus \Gamma_-$ denote the $\mathrm{su}(2)_+ \oplus \mathrm{su}(2)_-$ valued connection one-form of the Levi-Civita connection on $(M, g)$. Now the action of SO(4) on the orthonormal frame bundle induces an action of $\mathrm{SU}(2)_\pm$ on the locally defined spin bundles $V_\pm M$ which passes to an action on the projective bundles [**1**] $\mathbf{P}V_\pm M$, and we can view the fiber coordinates of $V_+ M$ as homogeneous coordinates along the fibers of the bundle $\mathbf{P}V_+ M \xrightarrow{\pi} M$ . There is a natural twisted one-form $\psi$ on $\mathbf{P}V_+ M$ determined by $\Gamma_+$ and the symplectic structure $\varepsilon$. In order to minimize the notational baggage we write a dot to denote the pairing induced by $\varepsilon$. We have

$$\psi = \varepsilon(z, dz + \Gamma \cdot z) = z \cdot dz + z \cdot \pi^* \Gamma_+ z.$$

Now if $(M, g)$ is an almost hyperhermitian structure the bundle $\mathbf{P}V_+ M$ is trivializable by the constant section $z$. In this case, $z^*\psi = z \cdot \Gamma_+ z =: \gamma$ and there is a two-sphere's worth of almost complex structures on $M$.

PROPOSITION 2. *$\gamma$ is type $(1,0)$ with respect to every almost complex structure $z \in \mathbf{P}^1$ on $M$ compatible with $g$ if and only if the almost hypercomplex structure on $M$ is integrable.*

PROOF. A basis for the type $(1,0)$ forms is given by the $V^-$-valued one-form $\eta = z \cdot H$. The complex structures determined by $z$ will all be integrable if and only if $d\eta$ has no $(0,2)$ component. So we compute using the first Cartan structure

equation

$$\begin{aligned} d\eta &= z \cdot dH = -z \cdot (\Gamma_+ \wedge H + \Gamma_- \wedge H) \\ &= -z \cdot \Gamma_+ \wedge \left( \eta \otimes \frac{\bar{z}}{|z|^2} + \bar{\eta} \otimes z \right) - \Gamma_- \wedge z \cdot H \\ &= -\frac{z}{|z|^2} \cdot \Gamma_+ \bar{z} \wedge \eta - \Gamma_- \wedge \eta - z \cdot \Gamma_+ z \wedge \bar{\eta}. \end{aligned}$$

So the $(0,2)$ component of $d\eta$ is determined by the $(0,1)$ component of $\gamma = z \cdot \Gamma_+ z$, and this proves the proposition. □

THEOREM 2. *If $(M,g)$ is hyperhermitian, then $(M,g)$ is anti-self-dual, so $\mathbf{P}V_+M$ is a complex manifold. Furthermore, there is a holomorphic fibration $\mathbf{P}V_+M \to \mathbf{P}^1$.*

PROOF. On $\mathbf{P}V_+M$ the almost complex structure induced by $g$ is given by the local basis of one-forms of type $(1,0)$, $\{\psi, \eta^1, \eta^2\}$. This complex structure is integrable if and only if the ideal $\mathcal{I}$ generated by $\{\psi, \eta^1, \eta^2\}$ is closed under exterior differentiation. It is not difficult to see that $d\eta$ is always in $\mathcal{I}$, but $d\psi$ is not. In fact, the component of $\bar{\eta}^1 \wedge \bar{\eta}^2$ in $d\psi$ is precisely $W_+$ [**1**]. Now according to Proposition 2, $\gamma$ is spanned by $\eta^1$, $\eta^2$ down on $M$, so $\pi^*\gamma$ is spanned by $\eta^1, \eta^2$ up on $\mathbf{P}V_+M$. Thus $\psi$ is a $(1,0)$ form on $\mathbf{P}V_+M$ and so $d\psi \in \mathcal{I}$. Hence, the almost complex structure is integrable and $M$ is anti-self-dual by the Atiyah-Hitchin-Singer theorem [**1**]. Furthermore, we consider the subbundle generated by $\psi - \pi^*\gamma = z \cdot dz$. This is a local section of the canonical bundle on $\mathbf{P}^1$ written in homogeneous coordinates. Thus $\mathbf{P}V_+M$ fibers holomorphically over $\mathbf{P}^1$. □

Now suppose $(M,g)$ is almost hyperhermitian. Since there is a reduction of the frame bundle to $\mathrm{SU}(2)_-$, the bundle $\Lambda_+M$ is trivial, and a choice of local coframe $H$ on $M$ provides a global framing $S_+$ of $\Lambda^2_+M \otimes \mathbb{C} \simeq S^2V_+M$, namely $S_+ = \{S^{11}, S^{12}, S^{22}\}$ with the real structure given by $\overline{S^{11}} = S^{22}$, $\overline{S^{12}} = -S^{12}$. Now given an almost complex structure $z \in \mathbf{P}^1$, we can write the hyperhermitian two-form as

$$\omega_z = -\frac{2i}{|z|^2} z \cdot S_+ \cdot \bar{z} = -\frac{i}{|z|^2} \eta \wedge \bar{\eta} \tag{2.1}$$

and the canonical $(2,0)$ form with respect to the almost complex structure $z$ as

$$\kappa_z = z \cdot S_+ \cdot z. \tag{2.2}$$

Then a basis of $\Lambda_+M \otimes \mathbb{C}$ compatible with the splitting (1.3) is given by $\{\omega_z, \kappa_z, \bar{\kappa}_z\}$.

PROPOSITION 3. *Let $(M,g,I_z)$ be almost hyperhermitian. Then $(M,g,I_z)$ is hyperhermitian if and only if there is a real smooth one-form $\beta$ on $M$ depending only on $g$ such that*

$$\begin{aligned} d\omega_z + \beta \wedge \omega_z &= 0, \\ d\kappa_z + \beta \wedge \kappa_z &= 0, \\ d\bar{\kappa}_z + \beta \wedge \bar{\kappa}_z &= 0. \end{aligned} \tag{2.3}$$

We first prove a lemma.

LEMMA 1. *$(M, g, I_z)$ is hyperhermitian if and only if there is a smooth one-form $\beta$ on $M$ depending only on $g$ such that $\gamma = -\beta^{\#} \rfloor \kappa_z$ where $\beta \to \beta^{\#}$ is the isomorphism $T^*M \simeq TM$ induced by the metric $g$.*

PROOF. If there is a smooth one-form $\beta$ independent of $z$ satisfying $\gamma = -\beta^{\#} \rfloor \kappa_z$, then $\gamma$ is clearly type $(1,0)$ for each $z \in \mathbf{P}^1$ so $(M, g, I_z)$ is integrable by Proposition 2. Conversely, if $(M, g, I_z)$ is integrable $\gamma$ is type $(1,0)$ for all $z \in \mathbf{P}^1$, and it follows from the nondegeneracy of $\kappa_z$ for each $z$ that $\gamma = -\beta_z^{\#} \rfloor \kappa_z$ for some smooth one-form $\beta_z$. To see that $\beta$ is independent of $z$, consider the su(2)-valued connection one-form $\Gamma_+$. This is clearly independent of $z$, and under the identification $\mathrm{su}(2) \otimes \mathbf{C} \simeq \Lambda_+^2 \otimes \mathbf{C} \simeq S^2 V_+^*$, $\Gamma_+$ can be viewed as a section of $S^2 V_+^* M \otimes V_+^* \otimes V_-^* M$ which decomposes as $S^3 V_+^* M \otimes V_-^* M \oplus V_+^* \otimes V_-^* M$. Moreover, one easily sees that $\gamma$ is type $(1,0)$ for all $z \in \mathbf{P}^1$ if and only if $\Gamma_+$ lies entirely in the $V_+^* \otimes V_-^*$ component, and this easily implies that $\beta$ is independent of $z$. $\square$

PROOF OF PROPOSITION 3. The first Cartan structure equation for the local coframe $H$ induces a structure equation on the framing $\{S^{AB} : A, B = 1, 2\}$ of $\Lambda_+ M \otimes \mathbf{C}$, viz.,

$$dS^{AB} + \Gamma_C^A \wedge S^{BC} + \Gamma_C^B \wedge S^{AC} = 0$$

(sum on repeated indices) where $(\Gamma_B^A) = \Gamma_+$. Applying the lemma and performing a short computation shows that $(M, g, I_z)$ is hyperhermitian if and only if

$$dS^{AB} + \beta \wedge S^{AB} = 0.$$

The equations (2.3) then follow from this and (2.1) and (2.2). $\square$

We shall be interested in hyperhermitian manifolds of a certain type. A hyperhermitian manifold $(M, g, I_z)$ is called *locally conformally hyper-Kähler* if there is an open cover $\{U_\alpha\}$ of $M$ and locally defined smooth functions $\sigma_\alpha : U_\alpha \to \mathbf{R}$ such that the local metric $e^{\sigma_\alpha}(g \mid U_\alpha)$ is hyper-Kähler on $U_\alpha$ for all $\alpha$. It is easy to see from Proposition 3 that a hyperhermitian four-manifold $(M, g, I_z)$ is locally conformally hyper-Kähler if and only if $\beta$ is closed. If $\beta$ is exact then $(M, g, I_z)$ is globally conformally hyper-Kähler, so that $(M, e^\sigma g, I_z)$ is hyper-Kähler for a smooth function $\sigma$ on $M$. In this case $\beta = 0$ with respect to the hyper-Kähler structure $(M, e^\sigma g, I_z)$, so $\kappa_z$ is closed and thus holomorphic. Thus the canonical line bundle $K_z$ is holomorphically trivial for each $z \in \mathbf{P}^1$. It follows from the Enriques-Kodaira classification [**2**] that the only compact complex surfaces admitting a hyper-Kähler structure are a torus with its flat metric and $K3$ with a Yau metric [**5**].

**3. Proof of the Main Theorem.** The proof of the theorem is based on several observations which have been made in the context of anti-self-dual Hermitian surfaces [**4**]. First, if a compact complex surface admits a hyperhermitian structure the associated one-form $\beta$ must be closed. Second, if the first Betti number $b_1$ is even, $\beta$ must be exact, so $(M, g, I_z)$ must be conformally equivalent to a hyper-Kähler structure. Third, if $b_1$ is odd, $(M, g, I_z)$ is conformally equivalent to a metric with positive scalar curvature almost everywhere. In what follows we assume that $(M, g, I_z)$ is a compact hyperhermitian manifold of real dimension four.

LEMMA 2. *Let $(M, g, I_z)$ be a compact hyperhermitian four-manifold. Then $(M, g, I_z)$ is locally conformally hyper-Kähler.*

PROOF. Since $(M, g, I_z)$ is hyperhermitian, there is a one-form $\beta$ on $M$ satisfying (2.3) by Proposition 3. Taking the exterior derivative of (2.3) gives $d\beta \wedge \omega_z = 0$,

$d\beta\wedge\kappa_z = 0$, $d\beta\wedge\bar{\kappa}_z = 0$. The last two of these equations imply that $d\beta$ is type $(1,1)$ with respect to all complex structures $z \in \mathbf{P}^1$, and by the first of these equations and (1.3), $d\beta$ is anti-self-dual and thus harmonic. But by Hodge theory every exact harmonic form on a compact manifold must vanish, i.e. $\beta$ is closed. Thus $(M, g, I_z)$ is locally conformally hyper-Kähler. □

LEMMA 3. *Suppose $b_1$ is even. Then $(M, g, I_z)$ is conformally equivalent to a hyper-Kähler manifold.*

PROOF. When $b_1$ is even we have the Hodge decomposition [**2**], $H^1(M,\mathbb{C}) \simeq H_z^{1,0}(M,\mathbb{C}) \oplus H_z^{0,1}(M,\mathbb{C})$ for each complex structure $z \in \mathbf{P}^1$. Since $(M, g, I_z)$ is compact, $\beta$ is closed, by the previous lemma, so for each complex structure $z \in \mathbf{P}^1$ there are a holomorphic one-form $\alpha_z$ and a smooth function $\psi$ on $M$ such that $\beta = \alpha_z + \bar{\alpha}_z + d\psi$. Thus after the conformal transformation $g \to e^{\psi} g$, $\beta' := \alpha_z + \bar{\alpha}_z$ satisfies

$$d\beta'^C = i\bar{\partial}_z\alpha_z - i\partial_z\bar{\alpha}_z = 0,$$

where $\beta'^C = i(\alpha_z - \bar{\alpha}_z)$. But a straightforward computation of the divergence (with respect to the metric $g' = e^{\psi} g$) of $\beta'$ shows

$$\begin{aligned} -\delta'\beta' &= -*' d *' \beta' = -*' d(\beta'^C \wedge \Omega') = +*' \beta'^C \wedge d\Omega' = -*' \beta'^C \wedge \beta' \wedge \Omega' \\ &= *'\beta' \wedge \beta'^C \wedge \Omega' = \|\beta'\|^2, \end{aligned}$$

where $\| \ \|$ is the norm on $\Lambda^1 M$ induced by the metric $g'$. Integrating this over $M$ implies $\beta' = 0$ by the Bochner-Green theorem [**19**]. So $\beta$ is exact and thus conformally equivalent to a hyper-Kähler metric. □

LEMMA 4. *Suppose $b_1$ is odd. Then $(M, g, I_z)$ is conformally equivalent to a coordinate quaternionic Hopf surface.*

PROOF. First we show that $(M, g)$ is conformally equivalent to a metric with positive scalar curvature almost everywhere. Now since $(M, g, I_z)$ is anti-self-dual, a curvature computation [**4**, **16**] gives

$$R/3 = \tfrac{1}{2}\|\beta\|^2 - \delta\beta.$$

So by a theorem of Gauduchon [**8**] there is a conformal transformation so that $\delta\beta = 0$. Thus $R \geq 0$ and vanishes only at points where $\beta$ vanishes. But from Lemma 1, $d\beta = 0$, so $\beta$ is harmonic and its zero set must have measure zero, i.e. $R$ is positive almost everywhere. Now since $M$ is a spin manifold, it follows from a well-known theorem of Lichnerowicz [**13**] that the $A$-roof genus vanishes, so the Hirzebruch signature vanishes. But then since $W_+ = 0$, $(M, g, I_z)$ is conformally flat. Since $(M, g, I_z)$ is both conformally flat and locally conformally Kähler, it follows from a theorem of Vaisman [**17**, Theorem 2.2] that $(M, g, I_z)$ is a Hopf surface with its standard locally conformally flat structure.

We now show that $M$ must be a coordinate quaternionic Hopf surface given by Kato [**10**]. By definition of Hopf surface $M \simeq \mathbb{C}^2\backslash\{0\}/G$, where $G$ is a group of biholomorphic maps acting freely and properly discontinuously on $\mathbb{C}^2\backslash\{0\}$. Now since $M$ is hypercomplex and $\mathbb{C}^2\backslash\{0\} \to M$ is an unbranched cover, the hypercomplex structure on $M$ descends from the standard hypercomplex structure $I_z$ on $\mathbb{C}^2\backslash\{0\}$. Thus if $\Phi_g\colon \mathbb{C}^2\backslash\{0\} \to \mathbb{C}^2\backslash\{0\}$ denotes the action of $G$ on $\mathbb{C}^2\backslash\{0\}$ for a fixed $g \in G$, then $\Phi_g$ is a hypercomplex map, i.e. $\Phi_g \circ I_z = I_z \circ \Phi_g$ for all $g \in G$ and

$z \in \mathbf{P}^1$. So $\Phi_g$ must be quaternionic in the sense of Sommese for all $g \in G$, and thus $M$ must be one of the coordinate quaternionic Hopf surfaces given on Kato's list [**10**, Proposition 8]. □

This completes the proof of Theorem 1. □

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, CLARKSON UNIVERSITY, POTSDAM, NEW YORK 13676