A NOTE CONCERNING THE 3-MANIFOLDS WHICH SPAN CERTAIN SURFACES IN THE 4-BALL

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ABSTRACT. We consider surfaces of the form $F \cup_K D$ where $F$ is a Seifert surface and $D$ is a slicing disk for the knot $K$. We show that, in general, there is no 3-manifold $M$ which spans $F \cup_K D$ in the 4-ball such that $F$ can be compressed to a disk in $M$.

This note is concerned with the following problem. Suppose $K$ is a classical knot which is spanned by the Seifert surface $F$ in $S^3 = \partial B^4$ and the slice disk $D$ in $B^4$. Must the surface $F \cup_K D$ bound a smoothly embedded, compact, orientable 3-manifold $M$ (also referred to as a Seifert manifold) in which $F$ is compressible to a disk?

This problem arises quite naturally from the proof that slice implies algebraically slice. Indeed, the proof that slice implies algebraically slice boils down to verifying that given any $M$ as above (such an $M$ always exists by transversality) a 1/2-basis of $H_1(F; \mathbb{Q})$ can be realized by curves in $F$ which are rationally null-homologous in $M$. The above problem is equivalent, by Stallings' version of the Loop Theorem [S], to asking if an $M$ can be found in which a 1/2-basis for $H_1(F)$ is realized by curves in $F$ which are null-homotopic in $M$.

We establish in §2 that the answer to this problem is in general no. In fact, by appropriately modifying Zeeman's [Z] notion of twist spinning an arc, a collection of surfaces of the form $F \cup_K D$ are constructed in §1 for which we give necessary and sufficient conditions on $F \cup_K D$ in order for it to bound an $M$ in which $F$ can be compressed to a disk. These conditions are geometric in nature—yet, they are applicable.

1. The examples. This section is devoted to the construction of particular surfaces of the form $F \cup_K D$. The pertinent properties of these surfaces will be postponed until the next section.

We begin the construction of the desired surfaces with the following data:

1. $K'$ is a smooth knot in $S^3$,
2. $F'$ is an incompressible Seifert surface for $K'$,
3. $N_i(K')$, $i = 1, 2$, denote tubular neighborhoods of $K'$ where $N_1(K') \subset \text{int} N_2(K')$, and
4. $n$ is a positive integer.

Now let $K' = A \cup B$ where $A$ and $B$ are compact, connected subarcs of $K'$ satisfying $(\text{int} A) \cap (\text{int} B) = \emptyset$. We write $N_1(K')$ as the obvious decomposition $N(A) \cup N(B)$; see Figure 1. Note that $\Delta^4 = [S^3 - \text{int} N(B)] \times I$ is the 4-ball.
Under this identification, $A \times I$ becomes our desired slicing disk $D$ and $K = \partial D$. Taking $N(D) = N(A) \times I$ we have

$$[S^3 - \text{int } N(K)] \times I = \text{closure } (\Delta^4 - N(D)).$$

To describe $F$, it suffices to show how $F$ meets $\partial \Delta^4 - \text{int } N(K)$. In fact, henceforth we will view $F$ as being properly embedded in $\partial \Delta^4 - \text{int } N(K)$ except when we write $F \cup K D$. To construct $F$ we begin by setting $F'' = F' \cap [S^3 - \text{int } N(K')]$. We presently construct a second surface, $G$, from $F''$ and the integer $n$ as follows. Take a parallel copy of $F''$ in $S^3 - \text{int } N(K)$. This parallel copy is now altered in $N_2(K') - \text{int } N_1(K')$ by twisting $\partial N_1(K')$ through almost $n$ full revolutions—so that now the boundary of the twisted copy agrees with $\partial F''$. The twisted copy is $G$; see Figure 2.

We define the desired surface $F$ by setting

$$F \cap ([S^3 - \text{int } N_1(K')] \times t) = \begin{cases} G, & t = 0, \\ \partial F'' \cap \partial N(B), & 0 < t < 1, \\ F'', & t = 1. \end{cases}$$

(Note that $\partial G = \partial F''$ so the above definition of $F$ makes sense.)

REMARK. As noted in the introduction, the surface $F \cup K D$ described above may also be constructed by means of Zeeman's twist spinning process. In this alternative construction, the slicing disk is (essentially) the disk in $B^4$ obtained by $n$ twist spinning the arc $A$ into its reflection. The corresponding Seifert surface is the boundary-connect-sum of $F''$ with its reflection. It is not difficult to show the resulting closed surface is the same as $F \cup K D$—the key step comes from noting what happens in $\partial B^4$ when the $n$ twist spun disk is ambient isotoped to the (untwisted) spun disk.
2. A theorem. In addition to the notation developed in §1 we will employ the following notation here. Let $X$ denote the $n$-fold cyclic cover of $S^3 - \text{int} N_1(K')$. Furthermore, let $F_*$ denote a parallel copy of $F''$ in $S^3 - \text{int} N_1(K')$ which lies between $F''$ and the parallel copy of $F''$ used in the construction of $G$; see Figure 3. $\tilde{F}'', \tilde{G}$, and $\tilde{F}_*$ are used to denote lifts of $F'', G$, and $F_*$, respectively, to $X$. These lifts are chosen subject to the properties (i) $(\text{int} \tilde{F}'') \cap (\text{int} \tilde{G}) = \emptyset$, equivalently $\tilde{F}'' \cup \tilde{G}$ is an embedded closed surface in $X$, and (ii) $\tilde{F}''$ and $\tilde{F}_*$ enjoy the same parallel relationship in $X$ as they do in $S^3 - \text{int} N_1(K')$; see Figure 4. Finally, let $U(\tilde{F}_*)$ denote the relative open "regular" neighborhood of $\tilde{F}_*$ in $X$ which corresponds to the component of $X - (\tilde{F}'' \cup \tilde{G})$ containing $\tilde{F}_*$.

The purpose of this section is to establish the

**Theorem.** The following are equivalent:

(I) $X - U(\tilde{F}_*)$ is a handlebody.

(II) $F \cup_K D$ bounds a smoothly embedded handlebody in $\Delta^4$.

(III) $F \cup_K D$ bounds a smoothly embedded, orientable, 3-manifold, $M$, in $\Delta^4$ such that $F$ can be compressed to a disk in $M$. 


There is a collection of loops in the kernel of the inclusion \( \pi_1(\text{int } F) \hookrightarrow \pi_1(\Delta^4 - N(D)) \) which realize a 1/2-basis of \( H_1(F) \).

REMARKS. (A) Throughout the statement of the Theorem the word ‘smooth’ should be taken as ‘smooth with corners’.

(B) In general the failure of (IV) would immediately imply that \( F \cup_K D \) cannot bound a 3-manifold in which \( F \) can be compressed to a disk. It should be noted however that we do not require in (IV) that the 1/2-basis be metabolic—which is a bit surprising initially.

Prior to entering into the proof of the Theorem we establish the following.

**LEMMA.** Let \( V \) denote an irreducible, orientable, compact, 3-manifold with \( \partial V \) being a connected surface of genus \( g \). Suppose a 1/2-basis of \( H_1(\partial V) \) is realized by curves in \( \partial V \) which are null-homotopic in \( V \). Then \( V \) is a handlebody.

**PROOF.** Let \( \alpha_1, \ldots, \alpha_g \) denote the curves in \( \partial V \) which represent a 1/2-basis of \( H_1(\partial V) \) and which are null-homotopic in \( V \). We use \( [\alpha_i] \) to denote the class represented by \( \alpha_i \) in \( H_1(\partial V) \). By Stallings’ version of the Loop Theorem, since \( [\alpha_i] \neq 0 \), there is a properly embedded disk \( D_i \subset V \) with \( [\partial D_i] \neq 0 \) in \( H_1(\partial V) \).

Inductively, we assume that there exists (individually) properly embedded disks \( D_1, \ldots, D_k \subset V \) with \( [\partial D_1], \ldots, [\partial D_k] \) being linearly independent over \( \mathbb{Z} \). Note then that there is a rank \( k \) submodule, \( N \), of \( H_1(\partial V) \) such that \( H_1(\partial V)/N \) is a free module and \( [\partial D_i] \in N \) for \( i = 1, \ldots, k \). If \( k < g \), we then have that there is some \( [\alpha_i] \notin N \). Again by applying Stallings’ version of the Loop Theorem there exists a properly embedded disk \( D_{k+1} \subset V \) with \( [\partial D_{k+1}] \notin N \). That \( [\partial D_1], \ldots, [\partial D_{k+1}] \) should be linearly independent now follows from the fact that \( H_1(\partial V)/N \) is free, hence \( \mathbb{Z} \cdot [\partial D_{k+1}] \cap N = 0 \). Inductively, there exist properly embedded disks \( D_1, \ldots, D_g \subset V \) with \( [\partial D_1], \ldots, [\partial D_g] \) being linearly independent over \( \mathbb{Z} \). Unfortunately, we might have these disks intersecting one another.

Suppose now without loss that all intersections of the \( D_i \) are transverse. Consider \( D_1 \cap D_2 \). If this intersection is nonempty, let \( A \) denote an outermost arc of intersection in \( D_1 \). Then \( A \) together with a subarc of \( \partial D_1 \) bounds a disk in \( D_1 \) whose interior misses \( D_2 \). (We can assume that \( D_1 \cap D_2 \) contains no circles of intersection by standard cut and paste arguments.) We may then cut \( D_2 \) along this disk to obtain \( D_{21} \) and \( D_{22} \), where each \( D_{2i} \) meets \( D_1 \) in at least one less arc than \( D_2 \). Evidently both \( D_{2i} \) are properly embedded by the outermost hypothesis on \( A \) and \( [\partial D_2] = [\partial D_{21}] + [\partial D_{22}] \). This last equality implies that at least one of the collections \( [\partial D_1], [\partial D_{21}], \ldots, [\partial D_g], i = 1, 2, \ldots, g \), must be linearly independent over \( \mathbb{Z} \). In this fashion, we may replace \( D_2 \) with \( D_2 \) such that \( D_2 \cap D_1 = \emptyset \) and \( [\partial D_1], [\partial D_{2}] \) are linearly independent over \( \mathbb{Z} \).

We proceed in this fashion with \( D_1 \cap D_3 \), etc., until all the disks can be assumed to miss \( D_1 \). We then make all the disks disjoint from \( D_2 \) by letting \( D_2 \) play the role that \( D_1 \) did above. Since \( \overline{D_2} \cap D_1 = \emptyset \), no new intersections of the higher indexed disks with \( D_1 \) will be created while cutting along subdisks of \( D_2 \). Evidently, proceeding in this manner we obtain \( g \) properly embedded, disjoint disks in \( V \) whose boundaries represent linearly independent elements of \( H_1(\partial V) \) over \( \mathbb{Z} \). In particular, these boundary curves cannot setwise separate \( \partial V \)—for this would imply they are linearly dependent over \( \mathbb{Z} \). It follows that upon compressing \( \partial V \) along these disks we are left with a 2-sphere. Since \( V \) is irreducible, this 2-sphere
bounds a 3-cell in $V$, which is the complement of the thickened 2-disks along which we have compressed $\partial V$. Hence $V$ is a handlebody. □

We now begin the

**Proof of the Theorem.** Initially, we note that (II) ⇒ (III) ⇒ (IV) are trivial implications.

(I) ⇒ (II). Let $\pi: X \to S^3 - \text{int } N_1(K')$ denote the covering projection. We now decompose $S^3 - \text{int } N_1(K')$ into possibly singular surfaces as follows: Initially we cut $S^3 - \text{int } N_1(K')$ along an open interval's worth of parallel copies of $F_*$, $F_* \times (-\varepsilon, \varepsilon)$ say. We then construct a Morse function

$$h: S^3 - [(\text{int } N_1(K') \cup (F_* \times (-\varepsilon, \varepsilon))] \to [-\varepsilon, \varepsilon].$$

Here we require $h^{-1}(\pm \varepsilon) = F \times \pm \varepsilon$, respectively, and the level sets of $h$ meet what remains of $\partial N_1(K')$ in parallel copies of $\partial F_*$. The level sets of $h$ together with $F_* \times (-\varepsilon, \varepsilon)$ stratify $S^3 - \text{int } N_1(K')$. This stratification of $S^3 - \text{int } N_1(K')$ lifts to a stratification of $X$. We now remove $F_* \times (-\varepsilon, \varepsilon)$ from $X$ and identify $\tilde{F}''$ with $\tilde{F}'' \times -\varepsilon$. Also, without altering the stratification outside $\pi^{-1}(N_2(K'))$, we twist $F_* \times \varepsilon$ into $G$. Under this twisting manipulation the boundary of each stratum is identified with $\partial \tilde{F}''$—so as to form a stratification of $X - U(\tilde{F}_*)$ which is pinched along the vertical boundary. We label these strata by $C_t$ where $t \in [0,1]$, $C_0 = \tilde{G}$, $C_1 = \tilde{F}''$, and $t \mapsto h \circ \pi(C_t \cap [X - \pi^{-1}(N_2(K'))])$ is a smooth, real-valued function. Note further that, if we have twisted $F_* \times \varepsilon$ into $\tilde{G}$ with due care, then $\pi|C_t$ is a diffeomorphism onto its image for all $t$. We now construct an embedding of $X - U(\tilde{F}_*)$ into the closure of $\Delta^4 - N(D)$ by requiring that the image of this embedding meets $(S^3 - \text{int } N_1(K')) \times t$ in the “stratum” $\pi(C_t)$ for all $t \in [0,1]$. This embedding immediately yields (I) ⇒ (II).

(IV) ⇒ (I). Let $p: (S^3 - \text{int } N_1(K')) \times I \to S^3 - \text{int } N_1(K')$ denote the canonical projection. By (IV), there exist loops $\lambda_1, \ldots, \lambda_g$ in $F$ which homotopically die in $(S^3 - \text{int } N_1(K')) \times I$ and realize a 1/2-basis of $H_1(F)$. The lifts of the loops $p(\lambda_1), \ldots, p(\lambda_g)$ to $\tilde{F}'' \cup \tilde{G}$ provide inessential loops $\alpha_1, \ldots, \alpha_g$ in $X$ which represent a 1/2-basis of $H_1(\tilde{F}'' \cup \tilde{G})$. Since $F'$ is an incompressible Seifert surface for $K'$, $F''$ is incompressible in $S^3 - \text{int } N_1(K')$ and therefore so must $F_*$ be incompressible in $S^3 - \text{int } N_1(K')$. It follows that $\tilde{F}_*$ is incompressible in $X$. The standard cut and paste argument now implies that the loops $\alpha_1, \ldots, \alpha_g$ are null-homotopic in $X - \tilde{F}_*$ and hence in $X - U(\tilde{F}_*)$. The lemma now applies to $X - U(\tilde{F}_*)$ to imply (I)—since Waldhausen [W] has shown that $X - U(\tilde{F}_*)$ is irreducible. □

**Remark.** Via the remark that concludes §1, the embedding constructed while verifying (I) ⇒ (II) may be identified with a portion of one fiber in Zeeman's fibering theorem; see Zeeman's Lemma 6. The remaining portion of this fiber is obtained by spinning $\tilde{F}''$ into its reflection—this spinning takes place in the complementary hemisphere of $S^4$.

Since many knots admit Seifert surfaces which are incompressible but have complements in $S^3$ other than open handlebodies (e.g. doubles of nontrivial knots) we obtain by taking $n = 1$

**Corollary 1.** There exist infinitely many surfaces of the form $F \cup_K D$ which do not admit a Seifert manifold in which $F$ can be compressed to a disk.
In light of the remark that concludes §1 we have

**COROLLARY 2.** The problem of whether a surface of the form \( F \cup_k D \) bounds a smoothly embedded, orientable 3-manifold \( M \) such that \( F \) is compressible to a disk in \( M \) depends on the choice of \( D \).

For this just note that if we obtain our slicing disk via spinning, rather than twist spinning, the resulting surface bounds a "spun" handlebody. (Readers familiar with Theorem 2 of [T] might note that Corollary 2 answers the natural disk dependence question there. In general, one must allow for stabilizations.)

3. **Concluding remarks.** That twist spinning should have such an adverse effect on the resulting surface bounding a handlebody should not be surprising, considering Zeeman’s fibering theorem. Of course, what is essential to the purposes of this paper is the fact that once \( F \) and \( D \) are identified along \( K \), one must consider the isotopy type of \( D \) (rel \( K \)).

A question arising from the Theorem is the following:

**QUESTION.** Does there exist a nonfibered knot \( K \) spanned by an incompressible Seifert surface \( F \) such that the 2-fold branched cyclic cover of \( S^3 \) over \( K \) yields a handlebody when cut along a lift of \( F \)?

The answer to this question might very well be known. I do not, however, know of an example, although there appears to be no group-theoretic reason for such an example not to exist.

It is perhaps reasonable to suggest that the techniques of the Theorem can be extended in a case by case fashion to determine whether a surface of the form \( F \cup_k D \) bounds a smooth 3-manifold in which \( F \) can be compressed to a disk provided \( D \) is a disk obtained by spinning. For instance, in [T] it is observed that if the initial \( K' \) is fibered then the resulting \( F \cup_k D \) bounds such a 3-manifold no matter which \( F \) is chosen. If \( D \) is not obtained by spinning the problem appears to be wide open—although the hypersurfaces constructed in Theorem 2 of [T] might be of use here.

**REFERENCES**


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