FUNCTION SPACES AND LOCAL CHARACTERS OF TOPOLOGICAL SPACES
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(Communicated by Dennis Burke)

ABSTRACT. We write $V \simeq W$ to mean that the two linear topological spaces $V$ and $W$ are linearly homeomorphic. In this paper we prove: (1) There are compact spaces $X, Y$ for which $C_p(X) \simeq C_p(Y)$ and $\chi(X) \neq \chi(Y)$ are satisfied. (2) For each infinite cardinal $\kappa$, there are spaces $X, Y$ for which $C_p(X) \simeq C_p(Y)$, $\chi(X) = \omega$ and $\psi(Y) = \kappa$. (3) For each infinite cardinal $\kappa$, there are spaces $X, Y$ for which $C_p(X) \simeq C_p(Y)$, $\pi_\chi(X) = \omega$ and $\pi_\chi(Y) = \kappa$.

All topological spaces considered here are Tychonoff spaces. For an arbitrary topological space $X$, the space $C_p(X)$ ($C_\sigma(X)$ in [2], $C_\pi(X)$ in [4]) is the set of all real-valued continuous functions on $X$ with the topology of pointwise convergence. The space $C_p(X)$ is a linear topological space under the algebraic operations being defined pointwise. We write $V \simeq W$ if two linear topological spaces $V$ and $W$ are linearly homeomorphic. Topological spaces $X, Y$ are said to be $l$-equivalent [1] if $C_p(X) \simeq C_p(Y)$. The weak dual of $C_p(X)$ is denoted by $L_p(X)$. It is well known that $C_p(X) \simeq C_p(Y)$ if and only if $L_p(X) \simeq L_p(Y)$ [5, 6]. Further, each topological space $X$ is embedded in $L_p(X)$ as a Hamel basis. Hence, if compact spaces $X$ and $Y$ are $l$-equivalent, then $\chi(Y) \leq |L_p(X)| \leq |\mathcal{P}(\mathcal{P}(X))| \leq 2^\omega \cdot 2^{\pi_\chi(X)}$.

In this note we will establish the following results.

EXAMPLE 1. There are $l$-equivalent compact spaces $X$ and $Y$ for which $\chi(X) = \omega$ and $\chi(Y) = 2^\omega$ are satisfied.

EXAMPLE 2. For each infinite cardinal $\kappa$, there are $l$-equivalent countably compact normal spaces $X$ and $Y$ for which $\chi(X) = \psi(X) = \omega$ and $\chi(Y) = \psi(Y) = \kappa$ are satisfied.

EXAMPLE 3. For each infinite cardinal $\kappa$, there are $l$-equivalent topological spaces $X$ and $Y$ for which $\pi_\chi(X) = \omega$ and $\pi_\chi(Y) = \kappa$ are satisfied.

Let $Y$ be a subspace of a topological space $X$. We define $C_p(X; Y) = \{f \in C_p(X): f(Y) = \{0\}\}$. Pavlovskii [5] showed the following lemma.

LEMMA 1. If $Y$ is a retract of a space $X$, then $C_p(X) \simeq C_p(Y) \times C_p(X; Y)$.

For a closed subset $Y$ of a normal space $X$, let $X/Y$ be the quotient space of $X$ obtained by collapsing $Y$ to one point $\ast$. In this case, the following lemma is a tedious exercise.

LEMMA 2. $C_p(X; Y) \simeq C_p(X/Y; \ast)$. 

Received by the editors December 30, 1985 and, in revised form, October 8, 1986.

1980 Mathematics Subject Classification (1985 Revision). Primary 54C30; Secondary 46E10.

Key words and phrases. Real-valued continuous function, linear homeomorphism, character, pseudocharacter, $\pi$-character.

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1. **Example 1.** Let us recall the Alexandroff double circle $C_1 \cup C_2$ [3]. The underlying set of this compact space is the union of two concentric circles $C_i = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 = i\}$, where $i = 1, 2$. The subspace $C_1$ is the unit circle $S^1$ with its natural topology and the subspace $C_2$ is a discrete space of cardinality $2^\omega$. Obviously $C_1$ is a retract of $C_1 \cup C_2$.

**CLAIM 1.** Let $A(2^\omega)$ be the one-point compactification of a discrete space of cardinality $2^\omega$. Then $C_p(S^1 \oplus A(2^\omega)) \simeq C_p(C_1 \cup C_2)$.

**PROOF.** Let $a$ be a point of $C_2$. Then
\[
C_p(C_1 \cup C_2) \simeq C_p(C_1 \cup (C_2 - \{a\})) \times C_p(\{a\}) \\
\simeq C_p(C_1) \times C_p(C_1 \cup (C_2 - \{a\}); C_1) \times R \\
\simeq C_p(C_1) \times C_p((C_1 \cup (C_2 - \{a\}))/C_1; *) \times R \\
\simeq C_p(S^1) \times C_p(A(2^\omega)) \simeq C_p(S^1 \oplus A(2^\omega)).
\]

Since $\chi(C_1 \cup C_2) = \omega$ and $\chi(S^1 \oplus A(2^\omega)) = 2^\omega$, it follows that the desired example is obtained.

2. **Example 2.** For each infinite cardinal $\kappa$, let $W(\kappa + 1)$ be the space of all ordinals less than $\kappa + 1$ with the usual interval topology. Let $V(\kappa + 1)$ be the subspace of $W(\kappa + 1)$ consisting of all ordinals whose cofinality is less than $\omega_1$.

**LEMMA 3.** Let $F(\kappa + 1)$ be the subspace of all limit points in $V(\kappa + 1)$. Then $F(\kappa + 1)$ is a retract of $V(\kappa + 1)$.

**PROOF.** For each $\alpha$ in $V(\kappa + 1)$, let
\[
r(\alpha) = \min\{\beta: \alpha \leq \beta, \beta \in F(\kappa + 1)\}.
\]

Then $r: V(\kappa + 1) \to F(\kappa + 1)$ is a retraction.

**CLAIM 2.** Let $A(\kappa)$ be the one-point compactification of a discrete space of cardinality $\kappa$. Then $C_p(A(\kappa) \oplus F(\kappa + 1)) \simeq C_p(V(\kappa + 1))$.

**PROOF.**
\[
C_p(V(\kappa + 1)) \simeq R \times C_p(V(\kappa + 1)) \\
\simeq R \times C_p(V(\kappa + 1); F(\kappa + 1)) \times C_p(F(\kappa + 1)) \\
\simeq R \times C_p(V(\kappa + 1)/F(\kappa + 1); *) \times C_p(F(\kappa + 1)) \\
\simeq C_p(V(\kappa + 1)/F(\kappa + 1)) \times C_p(F(\kappa + 1)) \\
\simeq C_p((V(\kappa + 1)/F(\kappa + 1)) \oplus F(\kappa + 1)).
\]

Since $V(\kappa + 1)$ is countably compact, the quotient space $V(\kappa + 1)/F(\kappa + 1)$ must be homeomorphic with the space $A(\kappa)$. Hence
\[
C_p(V(\kappa + 1)) \simeq C_p(A(\kappa) \oplus F(\kappa + 1)).
\]

It is obvious that
\[
\chi(V(\kappa + 1)) = \psi(V(\kappa + 1)) = \omega, \\
\chi(A(\kappa) \oplus F(\kappa + 1)) = \psi(A(\kappa) \oplus F(\kappa + 1)) = \kappa.
\]

3. **Example 3.** First consider regular cardinals. For each infinite regular cardinal $\lambda$, let $U(\lambda + 1)$ be the subspace of $W(\lambda + 1)$ consisting of all ordinals $\alpha$ such that $\text{cf}(\alpha) \leq \omega$ or $\alpha = \lambda$. 

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CLAIM 3. $C_p(U(\lambda + 1)) \simeq C_p(U(\lambda + 1)/\{\omega, \lambda\})$.

PROOF.

$C_p(U(\lambda + 1)) \simeq C_p(U(\lambda + 1); \{\omega, \lambda\}) \times C_p(\{\omega, \lambda\})$

$\simeq C_p(U(\lambda + 1)/\{\omega, \lambda\}; *) \times R^2$

$\simeq C_p(U(\lambda + 1)/\{\omega, \lambda\}) \times R$

$\simeq C_p(U(\lambda + 1)/\{\omega, \lambda\})$.

Note that $\pi_\chi(U(\lambda + 1)) = \lambda$, $\pi_\chi(U(\lambda + 1)/\{\omega, \lambda\}) = \omega$. Now consider a singular cardinal $\kappa$ and let $\{\lambda_\alpha : \alpha < \text{cf}(\kappa)\}$ be a transfinite sequence of regular cardinals such that

$$\sup\{\lambda_\alpha : \alpha < \text{cf}(\kappa)\} = \kappa.$$ 

Let $T(\kappa)$ be the topological sum of $\{U(\lambda_\alpha + 1) : \alpha < \text{cf}(\kappa)\}$, and let $S(\kappa)$ be the topological sum of $\{U(\lambda_\alpha + 1)/\{\omega, \lambda_\alpha\} : \alpha < \text{cf}(\kappa)\}$. Then, by Theorem 2.6 in [4],

$$C_p(T(\kappa)) \simeq C_p(S(\kappa)).$$

Further, it is obvious that $\pi_\chi(T(\kappa)) = \kappa$, $\pi_\chi(S(\kappa)) = \omega$.

BIBLIOGRAPHY


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