$L^p$ MULTIPLIERS; A NEW PROOF OF AN OLD THEOREM

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ABSTRACT. New proofs are given for the following results of Hirschman and Wainger: Let $\psi \in C^\infty(\mathbb{R}^n)$ vanish in a neighborhood of the origin; $\psi(\xi) = 1$ for large $\xi$. Then

$$|\xi|^{-\beta} \psi(\xi) \exp(|\xi|\alpha)$$

is a multiplier in $L^p(\mathbb{R}^n)$ for $|1/p - 1/2| < \beta/na$; is not a multiplier in $L^p(\mathbb{R}^n)$ for $|1/p - 1/2| > \beta/na$.

1. Introduction. Let $\alpha, \beta$ be real numbers, $0 < \alpha < 1$, $\beta > 0$. Let $\psi \in C^\infty(\mathbb{R}^n)$ be 0 near the origin and 1 outside a compact subset of $\mathbb{R}^n$. For $f \in S(\mathbb{R}^n)$ we define $Tf \in S(\mathbb{R}^n)$ by

$$[Tf]\sim(\xi) = |\xi|^{-\beta} \psi(\xi) \exp(i|\xi|\alpha)f(\xi).$$

$T$ is then a strongly singular convolution operator. The kernel of $T$ was first studied by G. H. Hardy for $n = 1$ (cf. [2]). In [3], I. I. Hirschman develops the $L^p$-theory for operators of type $T$, concentrating in the periodic case. He proves, for $n = 1$ (see [3, Theorem 3c and remarks following it]),

THEOREM 1. Let $|1/p - 1/2| < \beta/na$. Then $T$ extends to a bounded operator on $L^p$.

THEOREM 2. $T$ does not extend to a bounded operator on $L^p$ if $|1/p - 1/2| > \beta/na$.

The proof of Theorems 1, 2 for general dimension $n$ is due to S. Wainger [7] and E. Stein [5]. In [1], C. Fefferman picks up the problem by looking at the behavior of $T$ in the limit case $1/p - 1/2 = \beta/na$. He proves that in this case $T$ is somewhat better than of weak type $(p, p)$, thus proving Theorem 1 by the Marcinkiewicz interpolation theorem (cf., for example, [6]) and duality. His paper is also the first one to give information about behavior in the limit cases.

We will give new proofs of Theorems 1, 2 which we believe are simpler and more direct than the previous ones. In the sequel, $\| \cdot \|_p$ denotes the norm of $L^p(\mathbb{R}^n)$; $B(L^p)$ is the space of bounded operators on $L^p = L^p(\mathbb{R}^n)$; $\alpha, \beta, \psi$ are as described above; $0 < \delta < \rho$ are such that $\psi(\xi) = 0$ if $|\xi| \leq \delta$ and $\psi(\xi) = 1$ if $|\xi| \geq \rho$. All integrals without an explicit domain of integration are over all of $\mathbb{R}^n$.

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2. Proof of Theorem 1. We will need the following lemma.

**Lemma 1.** For $\sigma > 0$ define $k_\sigma$ on $\mathbb{R}^n$ by

$$k_\sigma(x) = (2\pi)^{-n/2} \int e^{ix\cdot \xi} \psi(\xi) \exp((i-\sigma)|\xi|^\alpha) \, d\xi.$$  

Then there exists a constant $C$ such that

$$\|k_\sigma\|_1 \leq C\sigma^{-n/2} \exp\left(-\frac{1}{4} \delta^\alpha \sigma\right)$$

for all $\sigma > 0$.

We postpone the proof of Lemma 1 to §4. Let $T$ be given by (1),

$$Tf(x) = (2\pi)^{-n/2} \int e^{ix\cdot \xi} \psi(\xi)|\xi|^{-\beta} \exp(i|\xi|^\alpha) \hat{f}(\xi) \, d\xi.$$  

Noticing that

$$|\xi|^{-\beta} = \frac{1}{\Gamma(\beta/\alpha)} \int_0^\infty \sigma^{(\beta/\alpha)-1} \exp(-\sigma|\xi|^\alpha) \, d\sigma,$$

we can write

$$Tf(x) = \frac{1}{\Gamma(\beta/\alpha)} \int_0^\infty \sigma^{(\beta/\alpha)-1} (k_\sigma * f)(x) \, d\sigma,$$

hence

$$\|Tf\|_p \leq \frac{1}{\Gamma(\beta/\alpha)} \int_0^\infty \sigma^{(\beta/\alpha)-1} \|k_\sigma * f\|_p \, d\sigma$$

for all $p \in [1, \infty]$. By Lemma 1,

$$\|k_\sigma * f\|_1 \leq C\sigma^{-n/2} \exp\left(-\frac{1}{4} \delta^\alpha \sigma\right)\|f\|_1;$$

since $|k_\sigma(\xi)| \leq \|\psi\|_\infty \exp\left(-\frac{1}{4} \delta^\alpha \sigma\right),$

$$\|k_\sigma * f\|_2 \leq \|\psi\|_\infty \exp\left(-\frac{1}{4} \delta^\alpha \sigma\right)\|f\|_2.$$

By the Riesz-Thorin interpolation theorem (cf. [8]) (5) and (6) imply for $1 \leq p \leq 2,$

$$\|k_\sigma * f\|_p \leq C_1\sigma^{-\lambda} \exp(-\nu\delta^\alpha \sigma)\|f\|_p$$

for $\sigma > 0$, where $C_1$ depends only on the $C$ of (5) and $\|\psi\|_\infty$, $\lambda = n((1/p) - (1/2))$ and $\nu = (1/p - 1/2)/2 + 2(1 - 1/p) > 0$. Using (7) in (4), we see $T \in B(L^p)$ if

$$\int_0^\infty \sigma^{(\beta/\alpha)-1-\lambda} \exp(-\nu\delta^\alpha \sigma) \, d\sigma < \infty,$$

which happens if (and only if) $\lambda < \beta/\alpha$; i.e., $1/p - 1/2 < \beta/n\alpha$. This proves Theorem 1 if $1 < p < 2$; the case $p \geq 2$ follows by duality.

3. Proof of Theorem 2. We use the following result from stationary phase analysis.
LEMMA 2. Let $P \in C^\infty(\mathbb{R}^n)$, $g \in C^\infty_c(\mathbb{R}^n)$ and assume $\det((\partial^2/\partial \xi_i \partial \xi_j)P(\xi)) \neq 0$ for all $\xi \in \text{supp } g$. Then there exists $C = C(g)$ such that

$$\left| \int \exp(ix \cdot \xi - itP(\xi))g(\xi) \, d\xi \right| \leq Ct^{-n/2}$$

for all $t > 0$.

For a proof of Lemma 2, see (for example) [4, p. 41].

For $t > 0$, we define

$$(8) \quad T(t)f(x) = (2\pi)^{-n/2} \int \exp(ix \cdot \xi + it|\xi|^\alpha)|\xi|^{-\beta} \psi(t^{1/\alpha} \xi) \hat{f}(\xi) \, d\xi$$

so that $T = T(1)$. Let $E$ be the set of all $f \in (\mathbb{R}^n)$ such that $\hat{f} \in C^\infty_0$ and $\text{supp } f \subseteq \{ \xi \mid |\xi| > \rho \}$. Noticing $\psi(t^{1/\alpha} \xi)\hat{f}(\xi) = \hat{f}(\xi)$ for $f \in E$, $t \geq 1$, all $\xi$; we can apply Lemma 2 with $P(\xi) = |\xi|^{\alpha}$ for $|\xi| \geq \rho$; $g(\xi) = |\xi|^{-\beta} \hat{f}(\xi)$. The determinant in the lemma evaluates to $(-\alpha|\xi|^{\alpha-2})(1-\alpha) \neq 0$ for $0 < \alpha < 1$, $|\xi| \geq \rho$. Thus

$$(9) \quad \|T(t)f\|_\infty \leq C(f)t^{-n/2}$$

for $f \in F$, $t \geq 1$. Assume now $1 \leq p \leq 2$ and $T \in B(L^p)$, of norm $C_p$. It is then easy to see $T(t) \in B(L^p)$ for all $t > 0$ and

$$(10) \quad \|T(t)f\|_p \leq C_p \|f\|_p$$

for all $f \in L^p$. From (10) and (9) we get at once

$$(11) \quad \|T(t)f\|_2 \leq Kt^\lambda$$

by a trivial interpolation, $K$ depending on $f$ and $\lambda = (p/2\alpha)[\beta/(n\alpha) - (1/p - 1/2)]$, $f \in E$. Since $\|T(t)f\|_2 = \|\hat{f}/|\xi|^{\alpha}\|_2$ for $t \geq 1$, i.e., is independent of $t$ for $t \geq 1$ and $f \in E \setminus \{0\}$, (11) can only hold if $\lambda \geq 0$, i.e., $1/p - 1/2 \leq \beta/na$. This proves Theorem 2 in case $1 \leq p \leq 2$; the case $p \geq 2$ follows by duality.

4. Proof of Lemma 1. For $r = 0, 1, 2, \ldots$, let $k^{(r)}(\sigma) = (\partial/\partial \sigma)^r k_\sigma$; thus

$$[k^{(r)}(\sigma)]^{-}(\xi) = (-1)^r \psi(\xi)|\xi|^{r\alpha} \exp((i - \sigma)|\xi|^{\alpha}).$$

If $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index of length $m$, then

$$(12) \quad |D_\xi^\gamma k^{(r)}(\sigma)(\xi)| \leq C[\chi_S(\xi) \exp(-\frac{1}{2}\delta^\alpha \sigma) + |\xi|^{r\alpha-m(1-\alpha)} \exp(-\frac{1}{2}|\xi|^{\alpha}\sigma)] \exp(-\frac{1}{4}|\xi|^{\alpha}\sigma)$$

for $|\xi| \geq \delta$, $\sigma > 0$, where $\chi_S$ is the characteristic function of $S = \{\xi \mid \delta \leq |\xi| \leq \rho \}$ and $C$ depends on $\psi$, $m$, $r$. The proof is straightforward, though tedious. Terms in which $\psi$ is differentiated can be estimated by const $\chi_S \exp(-|\xi|^{\alpha}\sigma)(1 + \sigma)^m$; in the other terms we can estimate (for $|\xi| \geq \delta$) all powers of $\xi$ by the highest appearing power, namely $|\xi|^{r\alpha-m(1-\alpha)}$. We get rid of all positive powers of $1 + \sigma$ by using part of $\exp(-|\xi|^{\alpha}\sigma) \leq \exp(-\sigma^{\alpha}\sigma)$ for that purpose. Another part of this factor gets factored out at the end. From (12) we get

$$(13) \quad \|x|^m k^{(r)}(\sigma)\|_2 \leq C \left[ \exp\left(-\frac{1}{2}\sigma^{\alpha}\sigma\right) + \sigma^{\lambda-(n/2\alpha)} \left( \int g(\xi) \, d\xi \right)^{1/2} \right] \exp\left(-\frac{1}{4}\sigma^{\alpha}\sigma\right)$$
where
\[ g(\xi) = |\xi|^{-2\lambda} \exp(-|\xi|^\lambda), \quad \lambda = \frac{m(1 - \alpha)}{\alpha} - r. \]

We see \( g \in L^1(\mathbb{R}^n) \) iff \( 2\lambda \alpha < n \), i.e.,
\begin{equation}
(14) \quad m < \frac{n + 2r\alpha}{2(1 - \alpha)}.
\end{equation}

Thus (13) implies
\begin{equation}
(15) \quad \| \, |x|^m k_{\sigma}^{(r)} \, \|_2 \leq C\sigma^{m(1-\alpha)/\alpha-(r+n/2\alpha)} \exp(-\frac{1}{4} \delta^\alpha \sigma)
\end{equation}
for all integers \( r, m \) such that (14) holds, \( \sigma > 0 \). In particular, (15) holds for \( m = 0, \) all \( r = 0, 1, \ldots \). Now choose and fix an integer \( m > n/2 \); then fix an integer \( r \geq 0 \) such that (14) holds. Using
\[ \|h\|_1 \leq \text{const} R^{n/2}(\|h\|_2 + R^{-m}\|\, |x|^m h \, \|_2), \]
valid for all \( R > 0 \), we get from (15) with \( h = k_{\sigma}^{(r)}, \) \( R = \sigma^{(1-\alpha)/\alpha}, \)
\[ \|k_{\sigma}^{(r)}\|_1 \leq C\sigma^{-(n/2)-r} \exp(-\frac{1}{4} \delta^\alpha \sigma) \]
for \( \sigma > 0 \). Integrating \( r \) times with respect to \( \sigma \), from \( \sigma \) to \( \infty \), we obtain (3).

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References

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