COMPLETENESS THEOREM FOR SINGULAR BIPROBABILITY MODELS

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ABSTRACT. The aim of the paper is to prove the completeness theorem for singular biprobability models. This also solves Keisler’s Problem 5.4 in the singular case (see [3]). The case of absolute continuity is considered in [6].

Let \( A \) be a countable admissible set and \( \omega \in A \). The logic \( L_{P_1P_2A} \) is similar to the standard probability logic \( L_{Ap} \). The only difference is that two types of probability quantifiers \( (P_1 \overrightarrow{x} \geq r \text{ and } P_2 \overrightarrow{x} \geq r) \) are allowed.

The probability logic \( L_{Ap} \) was introduced in [2] by H. J. Keisler. It is the logic which has formulas like those of \( L_A \subseteq L_{\omega_1\omega} \), except that quantifiers \( P \overrightarrow{x} \geq r \) (\( \overrightarrow{x} \) is a finite sequence of variables, \( r \in [0,1] \) a real number) are used instead of the usual \( \forall x \) and \( \exists x \). A model is a probability space with measurable relations with respect to some extension of product measure. A formula \( (P \overrightarrow{x} \geq r) \varphi(\overrightarrow{x}) \) is true in the model if \( \{ \overrightarrow{x} : \varphi(\overrightarrow{x}) \} \) has probability greater than \( r \).

A singular biprobability model is a structure \((A,\mu_1,\mu_2)\) where

\[
A = (A,R_i,c_j)_{i \in I,j \in J}
\]

is a classical structure without operations and \( \mu_1,\mu_2 \) are types of probability measures such that \( \mu_1 \) is singular with respect to \( \mu_2 \), i.e. \( \mu_1 \perp \mu_2 \).

The quantifiers are interpreted in the natural way i.e.

\[
(A,\mu_1,\mu_2) \models (P_i \overrightarrow{x} \geq r)\varphi(\overrightarrow{x}) \text{ iff }\mu_1^{(n)}(\{ \overrightarrow{x} \in A^n : (A,\mu_1,\mu_2) \models \varphi(\overrightarrow{x}) \}) \geq r
\]

for \( i = 1,2 \). (The measure \( \mu_i^{(n)} \) is the restriction of the completion of \( \mu_i^n \) to the \( \sigma \)-algebra generated by the measurable rectangles and the diagonal sets \( \{ \overrightarrow{x} \in A^n : x_i = x_j \} \).

Axioms and rules of inference are those of \( L_{Ap} \), as listed in [1] in conjunction with the axiom \( B_4 \) from [3] with the remark that both \( P_1 \) and \( P_2 \) can play the role of \( P \), together with the following axioms:

Axioms of continuity.

\[
\begin{align*}
(1) & \quad \bigwedge_{n \geq 1} \bigvee_{m \geq 1} \left( P_i \overrightarrow{y} < \frac{1}{n} \right) \left( P_j \overrightarrow{x} \in \left[ r - \frac{1}{m}, r \right] \right) \varphi(\overrightarrow{x},\overrightarrow{y}), \quad i,j = 1,2, \\
(2) & \quad \bigwedge_{n \geq 1} \bigvee_{m \geq 1} \left( P_i \overrightarrow{y} < \frac{1}{n} \right) \left( P_j \overrightarrow{x} \in \left( r, r + \frac{1}{m} \right] \right) \varphi(\overrightarrow{x},\overrightarrow{y}).
\end{align*}
\]

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Axioms of singularity.

\[(P_1 x = 0)((P_1 y > 0)(x = y) \land (P_2 y > 0)(x = y)), \quad i = 1, 2.\]

In order to prove the main result, let us introduce two sorts of auxiliary models.

**Definition.** (i) A weak structure for \(L_{P_1 P_2 A}\) is a structure \((\mathfrak{A}, \mu_{n,i})_{n \geq 1, i = 1, 2}\) (shortly \((\mathfrak{A}, \mu_{n,i})\)) such that each \(\mu_{n,i}\) is a finitely additive probability measure on \(A^n\) with each singleton measurable and the set \(\{b \in A^n: (\mathfrak{A}, \mu_{n,i}) \models \varphi \overline{a}, b\}\) is \(\mu_{n,i}\)-measurable for each \(\varphi(x, y) \in L_{P_1 P_2 A}\) and \(\overline{a} \in A^m\).

(ii) A middle structure for \(L_{P_1 P_2 A}\) is a weak structure \((\mathfrak{A}, \mu_{n,i})\) such that the following is true:

There is a set \(B \subseteq A\) such that \(\mu_1(B) = 0\) and \(\mu_2(B) = 1\).

In both cases the concept of satisfaction is defined in a natural way.

**Lemma 1.** A theory \(T\) of \(L_{P_1 P_2 A}\) is consistent if and only if it has a weak model in which each theorem of \(L_{P_1 P_2 A}\) is true.

The proof involves the Henkin construction similar to that in [1].

In order to prove the next lemma, we use the following Los-Marczewski theorem.

**Theorem 1** (see [4], [5]). Let \(C\) be a field of subsets of a set \(\Omega\). Let \(\mu\) be a finitely additive probability measure on \(C\). Let \(A \subseteq \Omega\) be such that \(A \notin C\). Let \(\mathcal{F}(C, A)\) be the smallest field on \(\Omega\) containing \(C\) and \(A\). Then there exists a finitely additive probability measure \(\bar{\mu}\) on \(\mathcal{F}(C, A)\) which is an extension of \(\mu\).

Moreover, if \(d\) is any real number between

\[
\mu_i(A) = \sup \left\{ \frac{\sum_{j=1}^{n} \mu(A_j) - \sum_{j=1}^{m} \mu(B_j)}{k I_A + \sum_{j=1}^{m} I_{B_j} \geq \sum_{j=1}^{n} I_{A_i}} : A_1, \ldots, A_n, B_1, \ldots, B_m, \right\}
\]

and

\[
\mu_e(A) = \inf \left\{ \frac{\sum_{j=1}^{n} \mu(A_j) - \sum_{j=1}^{m} \mu(B_j)}{k I_A + \sum_{j=1}^{m} I_{B_j} + k I_A} : A_1, \ldots, A_n, B_1, \ldots, B_m, \right\}
\]

then there is a finitely additive probability measure \(\bar{\mu}\) on \(\mathcal{F}(C, A)\) such that \(\bar{\mu}(A) = d\) and \(\bar{\mu}\) is an extension of \(\mu\) from \(C\) to \(\mathcal{F}(C, A)\). (\(I_A\) is an indicator function of the set \(A\)).

**Lemma 2.** A theory \(T\) of \(L_{P_1 P_2 A}\) is consistent if and only if it has a middle model in which each theorem of \(L_{P_1 P_2 A}\) is true.

**Proof.** The nontrivial part is to prove that if \(T\) is consistent then \(T\) has a middle model.

Let \((\mathfrak{A}, \mu_{n,i})\) be a weak model of \(T\) in which each theorem of \(L_{P_1 P_2 A}\) is true.

Let \(Z_0, Z_1, \ldots\) enumerate \(\text{dom} \mu_{1,1} = \text{dom} \mu_{1,2}\).
Let us choose sets $U$ and $S$ in the following way:

- $U_0 = \{ x \in A : \mu_{1,1}(\{x\}) > 0 \}$,
- $S_0 = \{ x \in A : \mu_{1,2}(\{x\}) > 0 \}$,
- $U_{n+1} = U_{n} \cup \{x\}$, where $x \in Z_n \setminus (U_n \cup S_n)$ (if there is any),
- $S_{n+1} = S_{n} \cup \{x\}$, where $x \in Z_n \setminus (U_{n+1} \cup S_n)$ (if there is any).

Let $U = \bigcup_{n \geq 0} U_n$ and $S = \bigcup_{n \geq 0} S_n$. It is easy to see that $U \cap S = \emptyset$ and that each set $Z_n \in \text{dom} \mu_{1,1} = \text{dom} \mu_{1,2}$ with a positive measure $\mu_{1,1}$ ($\mu_{1,2}$) has a nonempty intersection with $U$ ($U^c$).

Let us show that $(\mu_{1,1})_c(U) = 1$. (Let us abbreviate $(\mu_{1,1})_c$ as $\mu$ in the rest of the proof.)

Let $A_1, \ldots, A_n, B_1, \ldots, B_m$, be members from $\text{dom} \mu$ such that

$$\sum_{i=1}^{n} I_{A_i} \geq \sum_{i=1}^{m} I_{B_i} + k I_A.$$

Let $C_s = \{ x \in A : \sum_{i=1}^{n} I_{A_i}(x) = s \}$, $s = 0, \ldots, n$, $D_s = \{ x \in A : \sum_{i=1}^{m} I_{B_i}(x) = s \}$, $s = 0, \ldots, m$.

It is easy to see that $\mu(D_s) = 0$ for $s > n - k$ and $\mu(C_s) = 0$ for $s < k$. Consequently $\mu(D_{n-k}) + \cdots + \mu(D_{n-k-i}) \leq \mu(C_n) + \cdots + \mu(C_{n-i})$ for $i = 0, \ldots, n$, $k - 1$.

It follows that

$$\sum_{i=1}^{n} \mu(A_i) - \sum_{i=1}^{m} \mu(B_i) = \sum_{s=1}^{n} s \mu(C_s) - \sum_{s=1}^{m} s \mu(D_s)$$

$$= \sum_{s=k}^{n} s \mu(C_s) - \sum_{s=k}^{n-k} s \mu(D_s) = \sum_{s=k}^{n} s \mu(C_s) - \sum_{s=k}^{n} (s - k) \mu(D_{s-k})$$

$$\geq \sum_{s=k}^{n} s \mu(C_s) - \sum_{s=k}^{n} (s - k) \mu(C_s) = \sum_{s=k}^{n} k \mu(C_s) = k \sum_{s=0}^{n} \mu(C_s) = k.$$

So, $\mu_c(U) = 1$ or in precise notation $(\mu_{1,1})_c(U) = 1$.

Similarly it can be shown that $(\mu_{1,2})_c(U^c) = 1$.

Hence by Theorem 1, measures $\mu_{1,1}$ and $\mu_{1,2}$ can be extended so that $\mu_{1,1} \subseteq \bar{\mu}_{1,1}$, $\mu_{1,2} \subseteq \bar{\mu}_{1,2}$ and $\bar{\mu}_{1,1} \perp \bar{\mu}_{1,2}$.

**Lemma 3.** For each weak model $(\mathfrak{A}, \mu_{n,i})$ in which each theorem of $L_{P_1P_2A}$ is true there is a singular biprobability model $(\mathfrak{A}, \mu_1, \mu_2)$ such that $(\mathfrak{A}, \mu_{n,i}) \equiv (\mathfrak{A}, \mu_1, \mu_2)$.

The proof of the lemma makes use of Loeb-Hoover-Keisler construction (see [3]) and Lemma 2.

**Theorem 2 (Completeness Theorem for $L_{P_1P_2A}$ Logic).** A theory $T$ of $L_{P_1P_2A}$ is consistent if and only if $T$ has a singular biprobability model.

The proof follows easily from Lemmas 1, 2 and 3.
BIBLIOGRAPHY


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