COMPLETENESS THEOREM
FOR SINGULAR BIPROBABILITY MODELS

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ABSTRACT. The aim of the paper is to prove the completeness theorem for singular biprobability models. This also solves Keisler's Problem 5.4 in the singular case (see [3]). The case of absolute continuity is considered in [6].

Let $\mathcal{A}$ be a countable admissible set and $\omega \in \mathcal{A}$. The logic $L_{P_1, P_2, \mathcal{A}}$ is similar to the standard probability logic $L_{\mathcal{A}P}$. The only difference is that two types of probability quantifiers $(P_1 \overline{x} \geq r$ and $P_2 \overline{x} \geq r)$ are allowed.

The probability logic $L_{\mathcal{A}P}$ was introduced in [2] by H. J. Keisler. It is the logic which has formulas like those of $L_{\mathcal{A} \subseteq L_{\omega_1 \omega}}$, except that quantifiers $P \overline{x} \geq r$ (where $\overline{x}$ is a finite sequence of variables, $r \in [0, 1]$ a real number) are used instead of the usual $\forall x$ and $\exists x$. A model is a probability space with measurable relations with respect to some extension of product measure. A formula $(P \overline{x} \geq r) \varphi(\overline{x})$ is true in the model if $\{\overline{x} : \varphi(\overline{x})\}$ has probability greater than $r$.

A singular biprobability model is a structure $(\mathcal{A}, \mu_1, \mu_2)$ where

$$\mathcal{A} = (A, R_i, \ldots)_{i \in I, j \in J}$$

is a classical structure without operations and $\mu_1, \mu_2$ are types of probability measures such that $\mu_1$ is singular with respect to $\mu_2$, i.e. $\mu_1 \perp \mu_2$.

The quantifiers are interpreted in the natural way i.e.

$$(\mathcal{A}, \mu_1, \mu_2) \models (P \overline{x} \geq r) \varphi(\overline{x}) \text { if } \mu_1^{(n)} \{\overline{x} \in A^n : (\mathcal{A}, \mu_1, \mu_2) \models \varphi(\overline{x})\} \geq r$$

for $i = 1, 2$. (The measure $\mu_i^{(n)}$ is the restriction of the completion of $\mu_i^n$ to the $\sigma$-algebra generated by the measurable rectangles and the diagonal sets $\{\overline{x} \in A^n : x_i = x_j\}$.)

Axioms and rules of inference are those of $L_{\mathcal{A}P}$, as listed in [1] in conjunction with the axiom $B_4$ from [3] with the remark that both $P_1$ and $P_2$ can play the role of $P$, together with the following axioms:

**Axioms of continuity.**

1. $$\bigwedge_{n \geq 1} \bigvee_{m \geq 1} \left( P_i \overline{y} < \frac{1}{n} \right) \left( P_j \overline{x} \in \left[ r - \frac{1}{m}, r \right] \right) \varphi(\overline{x}, \overline{y}), \quad i, j = 1, 2,$$

2. $$\bigwedge_{n \geq 1} \bigvee_{m \geq 1} \left( P_i \overline{y} < \frac{1}{n} \right) \left( P_j \overline{x} \in \left( r, r + \frac{1}{m} \right] \right) \varphi(\overline{x}, \overline{y}).$$
Axioms of singularity.

\[(P_1x = 0)((P_1y > 0)(x = y) \land (P_2y > 0)(x = y)), \quad i = 1, 2.\]

In order to prove the main result, let us introduce two sorts of auxiliary models.

**Definition.** (i) A weak structure for \(L_{P_1P_2A}\) is a structure \((\mathfrak{A}, \mu_{n,i})_{n \geq 1, i = 1, 2}\) (shortly \((\mathfrak{A}, \mu_{n,i})\)) such that each \(\mu_{n,i}\) is a finitely additive probability measure on \(A^n\) with each singleton measurable and the set \(\{ \overline{b} \in A^n : (\mathfrak{A}, \mu_{n,i}) \models \varphi[\overline{a}, \overline{b}] \}\) is \(\mu_{n,i}\)-measurable for each \(\varphi(x, y) \in L_{P_1P_2A}\) and \(\overline{a} \in A^m\).

(ii) A middle structure for \(L_{P_1P_2A}\) is a weak structure \((\mathfrak{A}, \mu_{n,i})\) such that the following is true:

There is a set \(B \subseteq A\) such that \(\mu_1(B) = 0\) and \(\mu_2(B) = 1\).

In both cases the concept of satisfaction is defined in a natural way.

**Lemma 1.** A theory \(T\) of \(L_{P_1P_2A}\) is consistent if and only if it has a weak model in which each theorem of \(L_{P_1P_2A}\) is true.

The proof involves the Henkin construction similar to that in [1].

In order to prove the next lemma, we use the following Los-Marczewski theorem.

**Theorem 1** (see [4], [5]). Let \(C\) be a field of subsets of a set \(\Omega\). Let \(\mu\) be a finitely additive probability measure on \(C\). Let \(A \subseteq \Omega\) be such that \(A \notin C\). Let \(\mathcal{F}(C, A)\) be the smallest field on \(\Omega\) containing \(C\) and \(A\). Then there exists a finitely additive probability measure \(\bar{\mu}\) on \(\mathcal{F}(C, A)\) which is an extension of \(\mu\).

Moreover, if \(d\) is any real number between

\[
\mu_i(A) = \sup \left\{ \frac{\sum_{j=1}^{n} \mu(A_j) - \sum_{j=1}^{m} \mu(B_j)}{k} : A_1, \ldots, A_n, B_1, \ldots, B_m, k I_A + \sum_{j=1}^{m} I_{B_j} \geq \sum_{j=1}^{n} I_{A_i} \right\}
\]

and

\[
\mu_e(A) = \inf \left\{ \frac{\sum_{j=1}^{n} \mu(A_j) - \sum_{j=1}^{m} \mu(B_j)}{k} : A_1, \ldots, A_n, B_1, \ldots, B_m, \sum_{j=1}^{n} I_{A_j} \geq \sum_{j=1}^{m} I_{B_j} + k I_A \right\},
\]

then there is a finitely additive probability measure \(\bar{\mu}\) on \(\mathcal{F}(C, A)\) such that \(\bar{\mu}(A) = d\) and \(\bar{\mu}\) is an extension of \(\mu\) from \(C\) to \(\mathcal{F}(C, A)\). (\(I_A\) is an indicator function of the set \(A\)).

**Lemma 2.** A theory \(T\) of \(L_{P_1P_2A}\) is consistent if and only if it has a middle model in which each theorem of \(L_{P_1P_2A}\) is true.

**Proof.** The nontrivial part is to prove that if \(T\) is consistent then \(T\) has a middle model.

Let \((\mathfrak{A}, \mu_{n,i})\) be a weak model of \(T\) in which each theorem of \(L_{P_1P_2A}\) is true.

Let \(Z_0, Z_1, \ldots\) enumerate \(\text{dom} \mu_{1,1} = \text{dom} \mu_{1,2}\).
Let us choose sets $U$ and $S$ in the following way:

$$U_0 = \{x \in A: \mu_{1,1}(\{x\}) > 0\}, \quad S_0 = \{x \in A: \mu_{1,2}(\{x\}) > 0\},$$

$$U_{n+1} = U_n \cup \{x\}, \text{ where } x \in Z_n \setminus (U_n \cup S_n) \text{ (if there is any)},$$

$$S_{n+1} = S_n \cup \{x\}, \text{ where } x \in Z_n \setminus (U_{n+1} \cup S_n) \text{ (if there is any)}.$$

Let $U = \bigcup_{n \geq 0} U_n$ and $S = \bigcup_{n \geq 0} S_n$. It is easy to see that $U \cap S = \emptyset$ and that each set $Z_n \in \text{dom } \mu_{1,1} = \text{dom } \mu_{1,2}$ with a positive measure $\mu_{1,1}$ ($\mu_{1,2}$) has a nonempty intersection with $U \ (U^c)$.

Let us show that $(\mu_{1,1})_e(U) = 1$. (Let us abbreviate $(\mu_{1,1})_e$ as $\mu$ in the rest of the proof.)

Let $A_1, \ldots, A_n, B_1, \ldots B_m,$ be members from $\text{dom } \mu$ such that

$$\sum_{i=1}^n I_{A_i} \geq \sum_{i=1}^m I_{B_i} + kI_A.$$

Let $C_s = \{x \in A: \sum_{i=1}^n I_{A_i}(x) = s\}, \ s = 0, \ldots, n, \ D_s = \{x \in A: \sum_{i=1}^m I_{B_i}(x) = s\}, \ s = 0, \ldots, m.$

It is easy to see that $\mu(D_s) = 0$ for $s > n - k$ and $\mu(C_s) = 0$ for $s < k$. Consequently $\mu(D_{n-k}) + \cdots + \mu(D_{n-k-i}) \leq \mu(C_n) + \cdots + \mu(C_{n-i})$ for $i = 0, \ldots, n, \ k-1$.

It follows that

$$\sum_{i=1}^n \mu(A_i) - \sum_{i=1}^m \mu(B_i) = \sum_{s=1}^n s\mu(C_s) - \sum_{s=1}^m s\mu(D_s)$$

$$= \sum_{s=k}^n s\mu(C_s) - \sum_{s=1}^{n-k} s\mu(D_s) = \sum_{s=k}^n s\mu(C_s) - \sum_{s=k}^n (s-k)\mu(D_{s-k})$$

$$\geq \sum_{s=k}^n s\mu(C_s) - \sum_{s=k}^n (s-k)\mu(C_s) = \sum_{s=k}^n k\mu(C_s) = k \sum_{s=0}^n \mu(C_s) = k.$$

So, $\mu_e(U) = 1$ or in precise notation $(\mu_{1,1})_e(U) = 1$.

Similarly it can be shown that $(\mu_{1,2})_e(U^c) = 1$.

Hence by Theorem 1, measures $\mu_{1,1}$ and $\mu_{1,2}$ can be extended so that $\mu_{1,1} \subseteq \bar{\mu}_{1,1}$, $\mu_{1,2} \subseteq \bar{\mu}_{1,2}$ and $\bar{\mu}_{1,1} \perp \bar{\mu}_{1,2}$.

**Lemma 3.** For each weak model $(\mathfrak{A}, \mu_{n,i})$ in which each theorem of $L_{P_1P_2A}$ is true there is a singular biprobability model $(\mathfrak{A}, \mu_1, \mu_2)$ such that $(\mathfrak{A}, \mu_{n,i}) \equiv (\mathfrak{A}, \mu_1, \mu_2)$.

The proof of the lemma makes use of Loeb-Hoover-Keisler construction (see [3]) and Lemma 2.

**Theorem 2 (Completeness Theorem for $L_{P_1P_2A}$ Logic).** A theory $T$ of $L_{P_1P_2A}$ is consistent if and only if $T$ has a singular biprobability model.

The proof follows easily from Lemmas 1, 2 and 3.
BIBLIOGRAPHY


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