EXPANSION OF DISCRETE
AND CLOSURE-PRESERVING FAMILIES
TAKEMI MIZOKAMI
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ABSTRACT. In this paper, we define the classes of d-IP-expandable spaces and IP-expandable spaces, and study their properties and relations with orthocompact spaces and nonarchimedean quasi-metrizable spaces.

1. Introduction. Following [1], a space X is called CP-expandable if for each closure-preserving family \( \mathcal{F} = \{ F_\lambda : \lambda \in \Lambda \} \) of closed subsets of X and for each family \( \mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \} \) of open subsets of X such that \( F_\lambda \subseteq U_\lambda \) for each \( \lambda \), there exists a closure-preserving family \( \mathcal{V} = \{ V_\lambda : \lambda \in \Lambda \} \) of open subsets of X such that \( F_\lambda \subseteq V_\lambda \subseteq \overline{V_\lambda} \subseteq U_\lambda \) for each \( \lambda \). In this paper, we introduce the classes of IP-expandable spaces and d-IP-expandable spaces by replacing "closure-preserving" with other conditions. Our main purpose is to study the properties of these classes and the relations with orthocompact spaces and nonarchimedean quasi-metrizable spaces.

All spaces are assumed to be \( T_1 \) topological spaces and \( \mathbb{N} \) always denotes the set of natural numbers.

2. D-IP-expandability and IP-expandability. We state the definitions of d-IP-expandability and IP-expandability.

DEFINITION 2.1. We call a space X d-IP-expandable if for a discrete family \( \mathcal{F} = \{ F_\lambda : \lambda \in \Lambda \} \) of closed subsets of X and a family \( \mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \} \) of open subsets of X such that \( F_\lambda \subseteq U_\lambda \) for each \( \lambda \), there exists an interior-preserving family \( \mathcal{V} = \{ V_\lambda : \lambda \in \Lambda \} \) of open subsets of X such that \( F_\lambda \subseteq V_\lambda \subseteq U_\lambda \) for each \( \lambda \).

DEFINITION 2.2. We call a space X IP-expandable if for a closure-preserving family \( \mathcal{F} = \{ F_\lambda : \lambda \in \Lambda \} \) of closed subsets of X and a family \( \mathcal{U} = \{ U_\lambda : \lambda \in \Lambda \} \) of open subsets of X such that \( F_\lambda \subseteq U_\lambda \) for each \( \lambda \), there exists a family \( \mathcal{V} = \{ V_\lambda : \lambda \in \Lambda \} \) of open subsets of X such that \( F_\lambda \subseteq V_\lambda \subseteq U_\lambda \) for each \( \lambda \) and \( \{ V_\lambda : \lambda \in \Lambda \} \) is interior-preserving in X.

In either case, we call \( \mathcal{V} \) the IP-expansion of \( \mathcal{F} \) with respect to \( \mathcal{U} \) in X. A space X is called \( (\sigma-) \)orthocompact if every open cover of X has a \( (\sigma-) \)interior-preserving open refinement.

PROPOSITION 2.3. If a space X is collectionwise normal, then X is d-IP-expandable.

PROPOSITION 2.4. If a space X is orthocompact, then X is d-IP-expandable.

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Proof. Let $\mathcal{F} = \{F_\lambda: \lambda \in \Lambda\}$ and $\mathcal{U} = \{U_\lambda: \lambda \in \Lambda\}$ be the same pair of families as in Definition 2.1. Assume $F_\lambda \cap U_{\lambda'} = \emptyset$ if $\lambda \neq \lambda'$. Since $X$ is orthocompact, there exists an interior-preserving open refinement $\mathcal{W}$ of an open cover $\mathcal{U} \cup \{X - \bigcup \mathcal{F}\}$. Setting $V_\lambda = S(F_\lambda, \mathcal{W})$ for each $\lambda$, we have the IP-expansion $\mathcal{V} = \{V_\lambda: \lambda \in \Lambda\}$ of $\mathcal{F}$ with respect to $\mathcal{U}$.

In [5], Michael constructed a normal, noncollectionwise normal, metacompact space $X$. By Proposition 2.4, $X$ is $d$-IP-expandable. Thus, the converse of Proposition 2.3 is not true. Also, Scott constructed a countably compact space $X$ which is not orthocompact [8, Example 4.5]. Obviously, $X$ is $d$-IP-expandable. Hence the converse of Proposition 2.4 is also not true. The following gives a simple sufficient condition for a $d$-IP-expandable space to be orthocompact.

Theorem 2.5. If a space $X$ is submetacompact and $d$-IP-expandable, then $X$ is orthocompact.

Proof. Let $\mathcal{U}$ be an open cover of $X$. Since $X$ is submetacompact, that is $\theta$-refinable, by [11] there exists an open refinement $\bigcup_{n=1}^\infty \mathcal{U}_n$ of $\mathcal{U}$ and a closed cover $\{F_n: n \in \mathbb{N}\}$ of $X$ such that for each $n$, $\mathcal{U}_n$ covers $F_n$ and $\mathcal{U}_n$ is point-finite at each point of $F_n$. For each $n, k \in \mathbb{N}$, set the closed set by

$$E_{nk} = \{x \in F_n: \text{ord}(x, \mathcal{U}_n) \leq k\},$$

where $\text{ord}(x, \mathcal{U}_n) = |\{U \in \mathcal{U}_n: x \in U\}|$. Then $\{E_{nk}: n, k \in \mathbb{N}\}$ satisfies the following conditions

1. For each $n, k$, $\bigcup_{k=1}^\infty E_{nk} = F_n$ and $E_{nk} \subset E_{nk+1}$.
2. For each $n$, $E_{n1}$ is the union of a discrete family $\mathcal{E}_{n1}$ of closed subsets of $F_n$ such that each $E \in \mathcal{E}_{n1}$ is contained in some $U(E) \in \mathcal{U}_n$.
3. For each $n$ and each $k \geq 2$, if $T$ is a closed subset of $E_{nk}$ such that $T \cap E_{nk-1} = \emptyset$, then $T$ is the union of a discrete family $\mathcal{E}(T)$ of closed subsets of $E_{nk}$ such that each $E \in \mathcal{E}(T)$ is contained in some $U(E) \in \mathcal{U}_n$.

Let $n \in \mathbb{N}$ be fixed for a while. Since $X$ is $d$-IP-expandable, there exists the IP-expansion $\mathcal{V}_n$ of $\mathcal{E}_{n1}$ with respect to $\{U(E): E \in \mathcal{E}_{n1}\}$. By (3) and by $d$-IP-expandability of $X$ again, there exists the IP-expansion $\mathcal{V}_{n2}$ of $\mathcal{E}(E_{n2} - \bigcup \mathcal{V}_n)$ with respect to $\{U(E): E \in \mathcal{E}(E_{n2} - \bigcup \mathcal{V}_n)\}$. Repeating this process, we can get a sequence $\{V_{nk}: k \in \mathbb{N}\}$ of IP-expansions. It is easy to see that $\bigcup \{V_{nk}: n, k \in \mathbb{N}\}$ is a $\sigma$-interior-preserving open refinement of $\mathcal{U}$. By [3], $X$ is countably metacompact. Let $\{V_{nk}: n, k \in \mathbb{N}\}$ be a point-finite open refinement of $\{\bigcup V_{nk}: n, k \in \mathbb{N}\}$ such that $V_{nk} \subset \bigcup V_{nk}$ for each $n, k$. It is easy to see that

$$\bigcup \{V_{nk} \cap V: V \in V_{nk}\}: n, k \in \mathbb{N}$$

is an interior-preserving open refinement of $\mathcal{U}$.

The converse of Theorem 2.5 is not true, because there is a noncountably metacompact orthocompact space [8, Example 4.2]. A quasi-metric $d$ on a set $X$ with the property that $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, for each $x, y, z \in X$, is called nonarchimedean, and the space $(X, d)$ is called nonarchimedean quasi-metrizable. As is well known, nonarchimedean quasi-metrizable spaces are characterized as spaces that have a $\sigma$-interior-preserving base.
COROLLARY 2.6. Let $X$ be a developable space. Then $X$ is d-IP-expandable if and only if $X$ is nonarchimedean quasi-metrizable.

PROOF. Both if and only if parts follow easily from Theorem 2.5 and [4, Theorem 14].

It follows directly from the preceding result that every nonarchimedean quasi-metrizable space is $\sigma$-orthocompact, but the converse is not true [4, p. 116]. On the other hand, Sorgenfrey lines show that nonarchimedean quasi-metrizable spaces need not be developable. The following is not known:

QUESTION 2.7. If a space $X$ is developable and quasi-metrizable, then is $X$ d-IP-expandable?

This is equivalent to the well-known problem, due to Junnila, whether every developable quasi-metrizable space is nonarchimedean quasi-metrizable.

THEOREM 2.8. For a space $X$, the following are equivalent:

(1) $X$ is an orthocompact developable space.

(2) $X$ has a development $\{U_n : n \in \mathbb{N}\}$ such that each $U_n$ is interior-preserving in $X$.

(3) $X$ is a d-IP-expandable developable space.

(4) $X$ is a semistratifiable, nonarchimedean quasi-metrizable space.

PROOF. (1)$\rightarrow$(2) is trivial. (2)$\rightarrow$(3): Under (2), $X$ is a submetacompact $\sigma$-orthocompact space. Then $X$ is orthocompact. By Proposition 2.4, $X$ is d-IP-expandable. (3)$\rightarrow$(4) follows from Corollary 2.6. (4)$\rightarrow$(1): Under (4), $X$ is a submetacompact $\sigma$-orthocompact space, and therefore $X$ is orthocompact. Since a semistratifiable $\gamma$-space is developable [7], $X$ is developable.

COROLLARY 2.9. If for each $n \in \mathbb{N}$, $X_n$ is an orthocompact developable space, then so is $\prod_{n=1}^{\infty} X_n$.

PROOF. This follows from the fact that semistratifiability and having a $\sigma$-interior-preserving base are countably productive properties.

A space $X$ is said to have property (P) provided that for a closed $G_\delta$-set $F$ of $X$, there exists a family $\mathcal{U}$ of open subsets of $X$ satisfying the following:

(1) $\mathcal{U}/(X - F)$ is interior-preserving in $X - F$.

(2) For each open subset $V$ of $X$, there exists $U \in \mathcal{U}$ such that $V \cap F = U \cap F \subset U \subset V$.

THEOREM 2.10. If a space $X$ is nonarchimedean quasi-metrizable, then $X$ has the property (P).

PROOF. Write $F = \bigcap_{n=1}^{\infty} O_n$, where for each $n$, $O_n$ is open in $X$ and $O_{n+1} \subset O_n$. Let $\bigcup_{n=1}^{\infty} B_n$ be a base for $X$, where for each $n$, $B_n \subset B_{n+1}$ and $B_n$ is interior-preserving in $X$. Let $\{B(\lambda) : \lambda \in \Lambda\}$ be the totality of subfamilies of $\bigcup_{n=1}^{\infty} (B_n/O_n)$. Then it is easy to see that $\mathcal{U} = \{\bigcup B(\lambda) : \lambda \in \Lambda\}$ is the desired family.

COROLLARY 2.11. If a space $X$ is perfect and nonarchimedean quasi-metrizable then $X$ is d-IP-expandable.

PROOF. Let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ and $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be the same pair of families as in Definition 2.1. We apply the theorem to the closed subset $F = \bigcup\{F_\lambda : \lambda \in \Lambda\}$ to get a family $\mathcal{W}$ of open subsets of $X$ satisfying (1) and (2) above with $\mathcal{U}$ replaced by
Observe that for each $F \cup (U - F) = U$, is open in $X$ such that $F = U \cap F$. For each $\lambda$, take $W_\lambda \in \mathcal{W}$ such that

$$W_\lambda \cap F = F \subset W_\lambda \subset U_\lambda.'$$

Then it is easy to see that $\{W_\lambda: \lambda \in \Lambda\}$ is the IP-expansion of $\mathcal{F}$ with respect to $\mathcal{U}$.

We call a family $\mathcal{U}$ of open subsets of $X$ an outer base of a subset $F$ in $X$ if for each open subset $O$ with $F \subset O$ there exists $U \in \mathcal{U}$ such that $F \subset U \subset O$.

**COROLLARY 2.12.** If $X$ is perfect and nonarchimedean quasi-metrizable, then every closed subset $F$ of $X$ has an outer base $\mathcal{U}$ in $X$ such that $\mathcal{U}$ is interior-preserving in $X - F$.

**THEOREM 2.13.** Let $X$ be a developable space. Then $X$ is IP-expandable if and only if $X$ is d-IP-expandable.

**PROOF.** The "only if" part is trivial. "If" part: Let $\mathcal{F} = \{F_\lambda: \lambda \in \Lambda\}$ and $\mathcal{U} = \{U_\lambda: \lambda \in \Lambda\}$ be the same pair of families as in Definition 2.2. Since $X$ is semistatifiable, by the method of [10], we can get a family $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ of closed subsets of $X$ such that each $\mathcal{H}_n$ is discrete in $X$ and for each $\lambda$, there exists $\mathcal{H}(\lambda) \subset \mathcal{H}_n$ such that $F_\lambda = \bigcup \mathcal{H}(\lambda)$. Write $\mathcal{H}(\lambda) = \bigcup_{n=1}^{\infty} \mathcal{H}(\lambda, n)$, where $\mathcal{H}(\lambda, n) = \mathcal{H}(\lambda) \cap \mathcal{H}_n$ for each $n$. By Theorem 2.8, $X$ is nonarchimedean quasi-metrizable. Therefore, by Corollary 2.10, each $H \in \mathcal{H}$ has an outer base $\mathcal{U}(H)$ in $X$ such that $\mathcal{U}(H)$ is interior-preserving in $X - H$. For each $\lambda \in \Lambda$ and each $H \in \mathcal{H}(\lambda, n)$, $n \in \mathbb{N}$, we choose $U(H) \in \mathcal{U}(H)$ such that $U(H) \subset U_\lambda \cap O_n(F_\lambda)$. ({$O_n(F): n \in \mathbb{N}$} is the semistratification of $F$ in $X$.) Set

$$W_\lambda = \bigcup\{U(H): H \in \mathcal{H}(\lambda)\}.$$ 

Then it is easy to see that $F_\lambda \subset W_\lambda \subset U_\lambda$ for each $\lambda$. To see that $\{W_\lambda - F_\lambda: \lambda \in \Lambda\}$ is interior-preserving in $X$, let $p \in \bigcap\{W_\lambda - F_\lambda: \lambda \in \Lambda_0\}$ for $\Lambda_0 \subset \Lambda$. There exists $n \in \mathbb{N}$ such that $p \in X - O_n(\bigcup\{F_\lambda: \lambda \in \Lambda_0\})$. Since

$$\bigcup_{k=1}^{n-1} \bigcup\{U(H): H \in \bigcup\{\mathcal{H}(\lambda, k): \lambda \in \Lambda_0\}\},$$

is interior-preserving at $p$, we obtain an open set $O$ of $X$ such that $p \in O \subset \bigcap\{W_\lambda - F_\lambda: \lambda \in \Lambda_0\}$.

Nagami introduced the class of $L$-spaces, which lies between the classes of Lašnev spaces and $M_1$-spaces [6]. He called a space $X$ an $L$-space if $X$ is a paracompact $\sigma$-space such that each closed subset $F$ of $X$ has a closure-preserving outer base and at the same time has an outer base which is interior-preserving in $X - F$. From the definition, we easily have the following result.

**THEOREM 2.14.** Let $X$ be a stratifiable space. Then $X$ is an $L$-space if and only if $X$ is IP-expandable.

There exists a stratifiable space $X$ which is not an $L$-space [6, Example 2.2]. Therefore, d-IP-expandability need not imply IP-expandability even if $X$ is orthocompact.

Following [2], a space $X$ is called $D$-expandable if for any discrete family $\{F_\lambda: \lambda \in \Lambda\}$ of closed subsets of $X$ and each family $\{U_\lambda: \lambda \in \Lambda\}$ of open subsets of $X$ such
that $F_\lambda \subset U_\lambda$ for each $\lambda$ and $F_\lambda \cap U_\mu = \emptyset$ whenever $\lambda \neq \mu$, there exists a dissectable family $V = \{V_\lambda : \lambda \in \Lambda\}$ of open subsets of $X$ such that $F_\lambda \subset V_\lambda \subset U_\lambda$ for each $\lambda$. (For the definition of dissectable families, refer to [2].) Brandenburg showed that a space is $D$-paracompact if and only if it is submetacompact and $D$-expandable [2, Theorem 1].

**THEOREM 2.15.** If a space $X$ is semistratifiable, then $d$-$IP$-expandability implies $D$-expandability.

**PROOF.** It suffices to show that every interior-preserving family $U = \{U_\lambda : \lambda \in \Lambda\}$ of open subsets of a semistratifiable space $X$ is dissectable. Since $\{X - U_\lambda : \lambda \in \Lambda\}$ is a closure-preserving family of closed subsets of $X$, by the method of [10] there exists a family $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ of closed subsets of $X$ satisfying the following:

1. Each $\mathcal{H}_n$ is discrete in $X$.
2. For each subset $A_0 \subset A$, if $p \in \bigcap\{U_\lambda : \lambda \in A_0\}$ then $p \in H \subset \bigcap\{U_\lambda : \lambda \in A_0\}$ for some $H \in \mathcal{H}$.

For each $n \in \mathbb{N}$ and each $\lambda \in \Lambda$, set

$$H_{\lambda n} = \bigcup\{H \in H_n : H \subset U_\lambda\}.$$ 

Then by (2), $U_\lambda = \bigcup_{n=1}^{\infty} H_{\lambda n}$ for each $\lambda$. Since $U$ is interior-preserving in $X$, it is easy to see that $U$ is dissectable in $X$.

However, these notions of expandability are very different, because there exists a nonorthocompact developable space $X$ (for example, $X = (H_0, U)$ in [9, Example 4.9]). Therefore, $D$-expandability need not imply $d$-$IP$-expandability. Also, there exists a perfect subparacompact nonarchimedean quasi-metrizable space [2, Example 1], which is not $D$-paracompact. Therefore, the converse is also not true.

**REFERENCES**


**DEPARTMENT OF MATHEMATICS, JOETSU UNIVERSITY OF EDUCATION, JOETSU, NIIGATA 943, JAPAN**

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