TWO QUESTIONS ON HEEGAARD DIAGRAMS OF $S^3$

JOSE MARÍA MONTESINOS

(Communicated by Haynes R. Miller)

ABSTRACT. We review some of the methods that have been used to recognize $S^3$ from a Heegaard diagram. We propose a revision of these methods and examine their failure for manifolds different from $S^3$.

1. A Heegaard diagram of a closed, connected 3-manifold will be denoted by $(F; dv, dw)$, where $(M, F)$ is the underlying Heegaard splitting, i.e. $F$ is a closed, connected surface embedded in $M$, such that the closures of the two components of $M\setminus F$ are handlebodies $V$ and $W$, and where $v$ and $w$ are complete systems of meridians for $V$ and $W$. It is assumed also that $dv$ cuts $dw$ transversally.

A problem important for its relation with the Poincaré conjecture is to decide if a given Heegaard diagram corresponds to $S^3$ (see, for instance, [2]). Due to the fact that the Heegaard splittings of $S^3$ are canonical [11], this problem is reduced to finding if $(M, F)$ has a trivial handle by inspecting the diagram $(F; dv, dw)$. If $(F; dv, dw)$ has a cancelling pair, i.e. curves $dvi, dwj$ which cut each other in a single point, then $(M, F)$ has a trivial handle. But it is easy to show that the converse is not always true.

An important contribution to the study of Heegaard diagrams is due to Singer [8] who, among other things, proved that between two systems of meridian discs $v$ and $v'$ of a handlebody $V$, there exist a finite sequence of systems

$$v = v^0, v^1, \ldots, v^n = v'$$

where $v^{i+1}$ comes from $v^i$ by a single Singer move ("geometric T-transformation" in [10]), i.e. replacing a disc $x$ of the system $v^i$ by a disc contained in $V\setminus v^i$.

The problem of detecting a trivial handle was approached by Whitehead as follows [13]. Let $(F; dv, dw)$ be a diagram with $n$ cancelling pairs $(v_i, w_i)$, $i = 1, \ldots, n$, such that $\# v \cap (w_1 + \cdots + w_n) = n$, and let $(F; dv', dw)$ be obtained from $(F; dv, dw)$ by taking a new system $v'$ in $V$. Whitehead shows that it is possible to construct a sequence of systems $v' = v^0, v^1, \ldots, v^m$ such that $\# v^i \cap (w_1 + \cdots + w_n) < \# v^{i-1} \cap (w_1 + \cdots + w_n)$, $i = 1, \ldots, m$; and $w_1, \ldots, w_n$ together with $n$ discs of $v^m$ form $n$ cancelling pairs. The construction of such a $v^i$ is automatic, once a "cut-point of $(w_1, \ldots, w_n)$ with respect to $v^{i-1}$" is detected (see [13]), and this cut-point always exists, as Whitehead proves.\footnote{\textsuperscript{2}}

Received by the editors August 1, 1986.

1980 Mathematics Subject Classification (1985 Revision). Primary 57M40; Secondary 57N12, 57M12.

Supported by “Comité Conjunto Hispano-Norteamericano” and NSF Grant 8120790.

\footnote{1}{That there exists an algorithm to decide this has been announced by W. Haken in his address to the “Workshop on 3-manifolds” 16.I.1985, MSRI. It only remains to find a practical one (see “Abstracts from workshop on 3-manifolds” MSRI preprint #07312-85).}

\footnote{2}{A cut-point of the dual diagram was called a “wave” in [14].}

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0002-9939/88 $1.00 + $.25 per page

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But not every diagram of \((M, F)\) is of type \((F; \partial v', \partial w)\), because \(w\) can also be modified. If this happens, Whitehead’s approach fails in general, as Whitehead himself probably knew [13, p. 56]. However, for \(M = S^3\), where every diagram \((F; v, w)\) comes from one which has (genus of \(F\)) cancelling pairs, it was believed [14] that either there is a cut-point of \(w\) with respect to \(v\), or there is one of \(v\) with respect to \(w\). If this were the case, the problem of detecting \(S^3\) would be solved. This, amazingly, is true for the Heegaard diagrams of genus two of \(S^3\) [4, 6], but is false for higher genus (see [9, 7] and two unpublished examples of Ochiai). These examples, however, have cancelling pairs and, therefore, are reducible (though not by Whitehead’s procedure). It is natural to ask

**Question 1.** Are there Heegaard diagrams of \(S^3\) without “cut-points” and without cancelling-pairs?

2. Another approach to the problem is due to Haken [2], who, using results of Whitehead [12] and Zieschang [15] (see [10]), remarks that given \((F; \partial v, \partial w)\) there exists an algorithm to obtain a \(v'\) such that \(#\partial v' \cap \partial w \leq #\partial v'' \cap \partial w\) for every \(v''\). Once \(v'\) is found, the roles of \((v', w)\) are interchanged, and, again using the algorithm, one determines a \(w'\), etc., etc., ... until finally one gets a \((F; \partial v, \partial w)\) such that

\[
#\partial v' \cap \partial w' \geq #\partial v \cap \partial w \leq #\partial v' \cap \partial w
\]

for every \((F; \partial v', \partial w')\). A diagram such as \((F; \partial v, \partial w)\) was called pseudominimal in [1], and we have just said that one such can always be obtained.

Waldhausen [10] thought that if \((F; \partial v, \partial w)\) is pseudominimal and if \((M, F)\) has a trivial handle, then \((F; \partial v, \partial w)\) ought to have a cancelling pair. Unfortunately this is false (see [1 and 5]). A different, and easier, example is due to Haken [3] (see [16]). It is the diagram of genus 2 of \(L(13, 5)\) (Figure 1), that was found by realizing geometrically the group presentation

\[
Z_{13} = \langle a, b : a^3b^{-2} = a^2b^3 = 1 \rangle.
\]

**Figure 1**
The diagram is pseudominimal without cancelling pairs. However the algorithm mentioned at the beginning of this section, applied to $(F; \partial v_1, \partial w)$, where $\partial v_1$ is a single curve, gives $w'$ such that $\# \partial v_1 \cap \partial w' < \# \partial v_1 \cap \partial w$ and such that $\# \partial v_1 \cap \partial w' \leq \# \partial v_1 \cap \partial w''$ for every $w''$. Using this, we can sharpen the procedure proposed by Haken (the Haken algorithm) as follows:

1st step. Get $(F; \partial v, \partial w)$ pseudominimal.

2nd step. Using the algorithm just mentioned, minimize $(F; \partial v_i, \partial w_i)$ and $(F; \partial v_j, \partial w)$ for every $w_i$ and $v_j$. If $g$ is the genus of $F$, the final product of these two steps are $2g$ "diagrams" (one system having $g$ curves, and the other a single curve).

I thought that if $(M, F)$ has a trivial handle, at least one of these $2g$ "diagrams" would exhibit a cancelling pair. And, in fact, this is what happens with the example in [1] (see [5]) and for the example of Haken (in Figure 1, the curves $(\partial v_1, \partial w'_1)$ are a cancelling pair). However the following example shows that this is not true in general:

**EXAMPLE.** The diagram of Figure 2 is pseudominimal without a cancelling pair, but the underlying Heegaard splitting has genus 2. This can be proved by realizing the two Singer moves (in $v$ and $w$ respectively) sketched at the lower part of Figure 2. The manifold $M$ is the Seifert manifold which is the 2-fold covering of $S^3$ branched over the torus link $\{3, 9\}$. Realizing the 2nd step of the algorithm we
obtain six "diagrams," namely:

\[(F; \partial v, \partial w_1), (F; \partial v, \partial w_2), (F; \partial v' = \partial(v_1, v'_2, v_3), \partial w_3)\]
\[(F; \partial v_1, \partial w), (F; \partial v_2, \partial w), (F; \partial v_3, \partial w)\]

(Figure 3), and none of them has a cancelling pair.

But still one can ask

Question 2. Let \((S^3, F; v, w)\) be pseudominimal and let \(g\) be the genus of \(F\). Does any of the \(2g\) "diagrams," obtained from \((F; \partial v, \partial w_1), (F; \partial v, \partial w)\) by the Haken algorithm, have a cancelling pair?

REMARK. Lemma 3 (p. 793) of [12] implies that it is impossible to reduce \(\#\partial v_j \cap \partial w_i\) by a single Singer move applied to any one of the already minimized \(2g\) diagrams \((F; \partial v, \partial w_1), (F; \partial v, \partial w)\).

I thank Maite Lozano for drawing my attention to Lemma 3 of [12].

REFERENCES

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MATHEMATICAL SCIENCES RESEARCH INSTITUTE, BERKELEY, CALIFORNIA 94720
DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE ZARAGOZA, ZARAGOZA, SPAIN
Current address: Facultad de Matemáticas, Universidad Complutense, 28040 Madrid, Spain