

TWO QUESTIONS ON HEEGAARD DIAGRAMS OF S^3

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ABSTRACT. We review some of the methods that have been used to recognize S^3 from a Heegaard diagram. We propose a revision of these methods and examine their failure for manifolds different from S^3 .

1. A *Heegaard diagram* of a closed, connected 3-manifold will be denoted by $(F; \partial v, \partial w)$, where (M, F) is the *underlying Heegaard splitting*, i.e. F is a closed, connected surface embedded in M , such that the closures of the two components of $M \setminus F$ are handlebodies V and W , and where v and w are complete systems of meridians for V and W . It is assumed also that ∂v cuts ∂w transversally.

A problem important for its relation with the Poincaré conjecture is to decide if a given Heegaard diagram corresponds to S^3 (see, for instance, [2]).¹ Due to the fact that the Heegaard splittings of S^3 are canonical [11], this problem is reduced to finding if (M, F) has a *trivial handle* by inspecting the diagram $(F; \partial v, \partial w)$. If $(F; \partial v, \partial w)$ has a *cancelling pair*, i.e. curves $\partial v_i, \partial w_j$ which cut each other in a single point, then (M, F) has a trivial handle. But it is easy to show that the converse is not always true.

An important contribution to the study of Heegaard diagrams is due to Singer [8] who, among other things, proved that *between two systems of meridian discs v and v' of a handlebody V , there exist a finite sequence of systems*

$$v = v^0, v^1, \dots, v^n = v'$$

where v^{i+1} comes from v^i by a single Singer move ("geometric T -transformation" in [10]), i.e. replacing a disc x of the system v^i by a disc contained in $V \setminus v^i$.

The problem of detecting a trivial handle was approached by Whitehead as follows [13]. Let $(F; \partial v, \partial w)$ be a diagram with n cancelling pairs (v_i, w_i) , $i = 1, \dots, n$, such that $\#v \cap (w_1 + \dots + w_n) = n$, and let $(F; \partial v', \partial w)$ be obtained from $(F; \partial v, \partial w)$ by taking a new system v' in V . Whitehead shows that *it is possible to construct a sequence of systems $v' = v^0, v^1, \dots, v^m$ such that $\#v^i \cap (w_1 + \dots + w_n) < \#v^{i-1} \cap (w_1 + \dots + w_n)$, $i = 1, \dots, m$; and w_1, \dots, w_n together with n discs of v^m form n cancelling pairs.* The construction of such a v^i is automatic, once a "cut-point of (w_1, \dots, w_n) with respect to v^{i-1} " is detected (see [13]), and this cut-point always exists, as Whitehead proves.²

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¹That there exists an algorithm to decide this has been announced by W. Haken in his address to the "Workshop on 3-manifolds" 16.I.1985, MSRI. It only remains to find a *practical* one (see "Abstracts from workshop on 3-manifolds" MSRI preprint #07312-85).

²A cut-point of the dual diagram was called a "wave" in [14].

But not every diagram of (M, F) is of type $(F; \partial v', \partial w)$, because w can also be modified. If this happens, Whitehead's approach fails in general, as Whitehead himself probably knew [13, p. 56]. However, for $M = S^3$, where every diagram $(F; v, w)$ comes from one which has (genus of F) cancelling pairs, it was believed [14] that either there is a cut-point of w with respect to v , or there is one of v with respect to w . If this were the case, the problem of detecting S^3 would be solved. This, amazingly, is true for the Heegaard diagrams of genus two of S^3 [4, 6], but is false for higher genus (see [9, 7] and two unpublished examples of Ochiai). These examples, however, have cancelling pairs and, therefore, are reducible (though not by Whitehead's procedure). It is natural to ask

Question 1. Are there Heegaard diagrams of S^3 without "cut-points" and without cancelling-pairs?

2. Another approach to the problem is due to Haken [2], who, using results of Whitehead [12] and Zieschang [15] (see [10]), remarks that given $(F; \partial v, \partial w)$ there exists an algorithm to obtain a v' such that $\#\partial v' \cap \partial w \leq \#\partial v'' \cap \partial w$ for every v'' . Once v' is found, the roles of (v', w) are interchanged, and, again using the algorithm, one determines a w' , etc., etc., ... until finally one gets a $(F; \partial \hat{v}, \partial \hat{w})$ such that

$$\#\partial v' \cap \partial \hat{w} \geq \#\partial \hat{v} \cap \partial \hat{w} \leq \#\partial \hat{v} \cap \partial w'$$

for every $(F; \partial v', \partial w')$. A diagram such as $(F; \partial \hat{v}, \partial \hat{w})$ was called *pseudominimal* in [1], and we have just said that one such can always be obtained.

Waldhausen [10] thought that if $(F; \partial v, \partial w)$ is pseudominimal and if (M, F) has a trivial handle, then $(F; \partial v, \partial w)$ ought to have a cancelling pair. Unfortunately this is false (see [1 and 5]). A different, and easier, example is due to Haken [3] (see [16]). It is the diagram of genus 2 of $L(13, 5)$ (Figure 1), that was found by realizing geometrically the group presentation

$$\mathbb{Z}_{13} = |a, b: a^3 b^{-2} = a^2 b^3 = 1|.$$

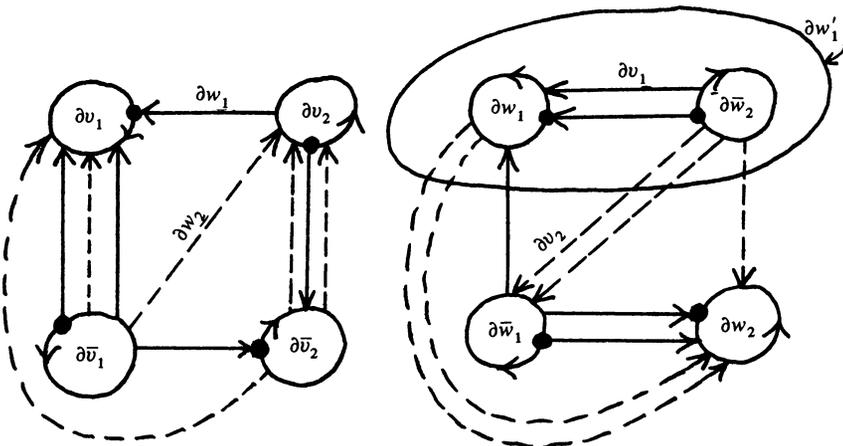


FIGURE 1

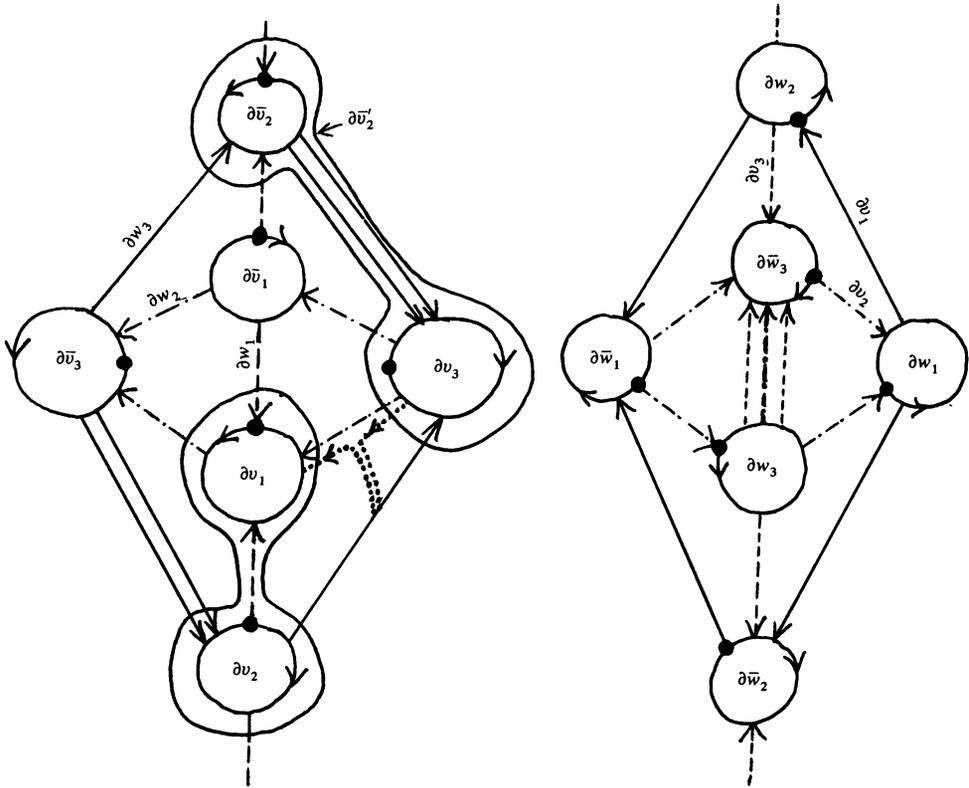


FIGURE 2

The diagram is pseudominimal without cancelling pairs. However the algorithm mentioned at the beginning of this section, applied to $(F; \partial v_1, \partial w)$, where ∂v_1 is a *single curve*, gives w' such that $\#\partial v_1 \cap \partial w' < \#\partial v_1 \cap \partial w$ and such that $\#\partial v_1 \cap \partial w' \leq \#\partial v_1 \cap \partial w''$ for every w'' . Using this, we can sharpen the procedure proposed by Haken (*the Haken algorithm*) as follows:

1st step. Get $(F; \partial v, \partial w)$ pseudominimal.

2nd step. Using the algorithm just mentioned, minimize $(F; \partial v, \partial w_i)$ and $(F; \partial v_j, \partial w)$ for every w_i and v_j . If g is the genus of F , the final product of these two steps are $2g$ “diagrams” (one system having g curves, and the other a single curve).

I thought that if (M, F) has a trivial handle, at least one of these $2g$ “diagrams” would exhibit a cancelling pair. And, in fact, this is what happens with the example in [1] (see [5]) and for the example of Haken (in Figure 1, the curves $(\partial v_1, \partial w'_1)$ are a cancelling pair). However the following example shows that this is not true in general:

EXAMPLE. The diagram of Figure 2 is pseudominimal without a cancelling pair, but the underlying Heegaard splitting has genus 2. This can be proved by realizing the two Singer moves (in v and w respectively) sketched at the lower part of Figure 2. The manifold M is the Seifert manifold which is the 2-fold covering of S^3 branched over the torus link $\{3, 9\}$. Realizing the 2nd step of the algorithm we

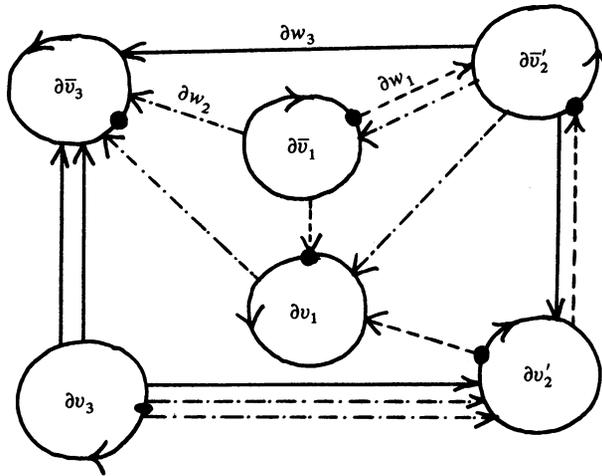


FIGURE 3

obtain six “diagrams,” namely:

$$(F; \partial v, \partial w_1), (F; \partial v, \partial w_2), (F; \partial v' = \partial(v_1, v'_2, v_3), \partial w_3) \\ (Figure\ 3), (F; \partial v_1, \partial w), (F; \partial v_2, \partial w), (F; \partial v_3, \partial w)$$

and none of them has a cancelling pair.

But still one can ask

Question 2. Let $(S^3, F; v, w)$ be pseudominimal and let g be the genus of F . Does any of the $2g$ “diagrams,” obtained from $(F; \partial v, \partial w_i)$, $(F; \partial v_j, \partial w)$ by the Haken algorithm, have a cancelling pair?

REMARK. Lemma 3 (p. 793) of [12] implies that it is impossible to reduce $\#\partial v_j \cap \partial w_i$ by a single Singer move applied to any one of the already minimized $2g$ diagrams $(F; \partial v, \partial w_i)$, $(F; \partial v_j, \partial w)$.

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