ON FORBIDDEN MINORS FOR $GF(3)$

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ABSTRACT. A new, surprisingly simple proof is given of the finiteness of the set of matroids minor-minimally not representable over $GF(3)$. It is, in fact, proved that every such matroid has rank or corank at most 3.

Introduction. For $q$ a prime power we denote by $L(q)$ the class of matroids representable over $GF(q)$, and by $F(q)$ the class of matroids which are minor-minimally not in $L(q)$. It was conjectured first by Rota [R] that $F(q)$ is finite for each $q$. This had been proved earlier by Tutte [T2] for $q = 2$, and has since been proved in a number of ways for $q = 3$ (R. Reid, unpublished circa 1972, or [S, B, T1, K]); but for larger $q$ the conjecture remains open.

Most of the above mentioned proofs proceed by actually determining the set $F(q)$, but, as remarked in [K1], nothing so precise is likely to be possible for general $q$. Our purpose here is to present a surprisingly simple proof of the finiteness of $F(3)$, specifically proving

THEOREM. If $M \in F(3)$ then either $M$ or $M^*$ has rank at most 3.

(Recall $M \in F(3) \iff M^* \in F(3)$.) Of course this reduces the precise determination of $F(3)$ to the determination of its rank 3 members, a problem easily settled by ad hoc arguments.

For matroid background we refer to Welsh [W], from which our notation differs in that we use $r(A)$ and $\bar{A}$ for the rank and closure of a set $A$.

1. A Lemma. To a large extent our proof parallels the easier part of the argument of [S]. The new idea, which eliminates virtually all of the hard work of that paper, is a timely application of the following simple fact.

LEMMA. Let $M$ be a connected simple matroid of rank $r \geq 2$ on a set $S$ and let

$$X = \{x \in S : M/x \text{ is disconnected}\}.$$

Then

(a) $|X| \leq r - 2$, and
(b) if $|X| = r - 2$ then there are lines $l_0, \ldots, l_{r-2}$ and an ordering $x_1, \ldots, x_{r-2}$ of $X$ such that

(i) $|l_i| \geq 3$,
(ii) $S = \bigcup_{i=0}^{r-2} l_i$, and
(iii) for $i \geq 1$, $l_i \cap (l_0 \cup \cdots \cup l_{i-1}) = \{x_i\}$.

(We remark that condition (i) in (b) is a consequence of conditions (ii) and (iii) and the connectivity of $M$.)

The proof will follow a few preliminaries. Let $M$ be as in the hypotheses of the Lemma. We define a *bisection* of $M$ at $x \in S$ to be a pair $(A, B)$ of flats of $M$ satisfying

$$A \cup B = S \neq A, B, \quad A \cap B = \{x\}$$

and

$$r(A) + r(B) = r(S) + 1.$$ 

If $M$ is connected the following are easily verified.

(1.1) If $(A, B)$ is a bisection of $M$ then $M|A$ and $M|B$ are connected.
(1.2) $M/x$ is disconnected if and only if there is a bisection of $M$ at $x$.
(1.3) If $(A, B)$ is a bisection then every circuit meeting both $A \setminus x$ and $B \setminus x$ is of the form $C \Delta D$ with $C \subseteq A$ and $D \subseteq B$ circuits containing $x$ (and $\Delta$ denoting symmetric difference).

Suppose now that $M$ is connected, that $(A, B)$ is a bisection at $x$, and that $(A', B')$ is a bisection at some $x' \in A \setminus B$. From (1.1) and (1.3) we have immediately

(1.4) $(A' \cap A, B' \cap A)$ is a bisection of $A$.
(Note e.g. $A' \cap A \neq \{x'\}$ since this would force $x' \in \bar{B}$.) Moreover
(1.5) only one of $A', B'$ meets $B$.

For suppose this is false. By (1.1) there is a circuit $C \subseteq B$ meeting both $A' \cap B$ and $B' \cap B$, and by (1.3) $C = C_1 \Delta C_2$ where $C_1 \subseteq A'$, $C_2 \subseteq B'$ and $C_1 \cap C_2 = \{x'\}$. But then $C_1 \setminus \{x'\} \subseteq C \subseteq B$ and this implies that $x'$ is in the span of $B$, a contradiction. 

**Proof of Lemma.** We proceed by induction on $r$, the case $r = 2$ being trivial.

Assume, then, that $r \geq 3$ and $|X| \geq r - 2$. Let $(A, B)$ be a bisection of $M$ with $A$ minimal at some $x \in X$. By (1.5) we must have

(1.6) $X \subseteq B$.

But now setting $M' = M|B$ and $X' = X \setminus \{x\}$ we apply (1.4) (with $A$ and $B$ reversed) and (1.2) to find that

(1.7) $M'/x'$ is disconnected for each $x' \in X'$.

But then since $M'$ is connected (by (1.1)) with $r(M') \leq r - 1$ and $|X'| \geq r - 3$, our inductive hypothesis implies that equality holds in both places (note this already proves (a)), that $A$ is a line, that

$$X' = \{x': M'/x' \text{ is disconnected}\},$$

and that we can find lines $l_0, \ldots, l_{r-3}$ of $M'$ and an ordering $x_1, \ldots, x_{r-3}$ of $X'$ satisfying the conditions of (b). The proof of (b) is now completed by taking $l_{r-2} = A$ and $x_{r-2} = x$. 

2. **Proof of the Theorem.** Suppose henceforth that $M$ is a rank $r$ matroid on $S$, $M \in \mathcal{F}(3)$.

It is easy to see that any member of any $\mathcal{F}(q)$ must be 3-connected, and so the matroid as well as all its single element deletions and contractions are connected. We recall Lemma 2.3 of [S].
(2.1) If $M$ is connected and $M\setminus x$ and $M/x$ are connected for each $x \in S$, then either $M = U_4^2$ or there exist $a, b \in S$ such that one of $M\setminus\{a, b\}$, $M/\{a, b\}$ is connected.

(As usual $U_4^2$ is the four element matroid whose circuits are its sets of size 3.) Since $U_4^2 \notin \mathcal{F}(3)$, and since our Theorem is self-dual, we may suppose that

(2.2) There exist $a, b \in S$ such that $M\setminus\{a, b\}$ is connected.

We attempt to represent $M$ over $GF(3)$ by starting with a representation

$$\phi: S\setminus\{a, b\} \to V = V_r(3)$$

of $M\setminus\{a, b\}$, and extending $\phi$ to $a$ and $b$ so that $\phi|_{S\setminus\{b\}}$ and $\phi|_{S\setminus\{a\}}$ are representations of $M\setminus b$ and $M\setminus a$. (This is possible since, as shown in [BL], $GF(3)$-representations are projectively unique.) Let $M'$ be the matroid on $S$ represented by $\phi$. Then $M$ and $M'$ are related as follows.

(2.3) $M$ and $M'$ are distinct connected matroids on a common set $S$,
(2.4) $M\setminus a = M'\setminus a$ and $M\setminus b = M'\setminus b$,
(2.5) $M\setminus\{a, b\}$ ($= M'\setminus\{a, b\}$) is connected.

PROPOSITION. If matroids $M$ and $M'$ satisfy (2.3)-(2.5) then at most one of them is in $\mathcal{L}(3)$.

PROOF. Suppose both are in $\mathcal{L}(3)$, and let $r$ denote their common rank. By the uniqueness of $GF(3)$-representations [BL] any representation

$$\phi: S\setminus\{a, b\} \to V$$

($V$ an $r$-dimensional $GF(3)$-vector space) of $M\setminus\{a, b\} = M'\setminus\{a, b\}$ can be extended to representations $\psi: S \to V$ and $\psi': S \to V$ of $M$ and $M'$ respectively. But up to scaling there is only one choice for $\psi(a)$ which makes $\psi|_{S\setminus\{b\}}$ a representation of $M\setminus b$, and similarly for $\psi(b)$. (This follows from the uniqueness of representations together with Kantor’s observation [K2, Lemma 7] that if $K$ is a field with no nonidentity automorphisms and $X$ a connected spanning subset of some projective space over $K$, then no nonidentity automorphism of the space fixes $X$ elementwise.) Of course the same discussion applies to $M'$, so from (2.4) we deduce that $\psi$ and $\psi'$ are equal up to scaling, and therefore $M$ and $M'$ are equal. □

In view of this result, our proof of the Theorem can be completed by showing

(2.6) If $M$ and $M'$ are a minor-minimal pair of matroids satisfying (2.3)-(2.5) then their (common) rank is at most 3.

PROOF. Let $Z$ be a minimal set which is dependent in one of $M$, $M'$ (say $M$) and independent in the other. Then as in the proof of Lemma 2.7 of [S], the minimality of $M$, $M'$ implies that $Z$ is unique and satisfies, among other things,

(2.7) $a, b \in Z$,
(2.8) $Z$ is a circuit and a hyperplane of $M$,
(2.9) For every $z \in Z\setminus\{a, b\}$, $M\setminus\{a, b\}/\{z\}$ is disconnected.

Now suppose $r \geq 4$. Since $|Z| = r$ (by (2.8)), (2.9) and the Lemma imply that there exist $x, y \in Z\setminus\{a, b\}$ (take $x, y$ to be $x_1, x_2$ as given by the Lemma) such that the closure of $\{x, y\}$ in $M$ contains some third element. But this contradicts (2.8) and we conclude that $r \leq 3$. This completes the proof of the Theorem. □
References


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