ANALYTIC AND DIFFERENTIABLE FUNCTIONS VANISHING ON AN ALGEBRAIC SET

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ABSTRACT. Let $U$ be an open semi-algebraic subset of $\mathbb{R}^n$ and let $X$ be a closed analytic subset of $U$ which is also a semi-algebraic set (e.g., $U = \mathbb{R}^n$ and $X$ is an algebraic subset of $\mathbb{R}^n$). It is proved that the ideal of analytic functions on $U$ vanishing on $X$ is finitely generated provided that the set $X$ is coherent. The ideal of infinitely differentiable functions on $U$ vanishing on $X$ is finitely generated if and only if the set $X$ is coherent.

1. The results. Let $U$ be an open subset of $\mathbb{R}^n$. Denote by $\mathcal{O}(U)$ and $\mathcal{E}(U)$ the ring of analytic functions and $C^\infty$ functions on $U$, respectively. Given a subset $X$ of $U$, we define $I(X)$ to be the ideal of $\mathcal{O}(U)$ and $I^\ast(X)$ to be the ideal of $\mathcal{E}(U)$ of all functions vanishing on $X$. It would be interesting to know under what assumptions on $X$ the ideals $I(X)$ and $I^\ast(X)$ are finitely generated (cf. [1, 4]). For instance, it remains an open question whether the ideal $I(X)$ is finitely generated if $X$ is an algebraic subset of $\mathbb{R}^n$ [4, p. 65].

We propose one result in this direction. Let $\mathcal{O}$ be the sheaf of germs of analytic functions on $\mathbb{R}^n$ and let $J(X)$ be the subsheaf of ideals of $\mathcal{O}|U$ of germs vanishing on $X$. Recall that if $X$ is a closed analytic subset of $U$, then $X$ is said to be coherent if the sheaf $J(X)$ is coherent. Denote by $\mathcal{E}$ the sheaf of germs of $C^\infty$ functions on $\mathbb{R}^n$ and by $J^\ast(X)$ the subsheaf of ideals of $\mathcal{E}|U$ of germs vanishing on $X$. We shall need the following characterization of coherent analytic sets.

**Proposition 1.** Let $U$ be an open subset of $\mathbb{R}^n$ and let $X$ be a closed analytic subset of $U$. Then $X$ is coherent if and only if for each $x$ in $U$ the stalk $J(x)$ is finitely generated over $\mathcal{E}_x$.

Let us recall that a subset of $\mathbb{R}^n$ is said to be semi-algebraic if it belongs to the smallest family of subsets which contains sets of the form $\{x \in \mathbb{R}^n | p(x) > 0\}$, where $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function, and which is closed under the Boolean operations of finite union, finite intersection and taking complements. Obviously, every algebraic subset of $\mathbb{R}^n$ is semi-algebraic.

**Theorem 2.** Let $U$ be an open semi-algebraic subset of $\mathbb{R}^n$ and let $X$ be a closed analytic subset of $U$ which is also a semi-algebraic set. Then:

(i) The ideal $I(X)$ is finitely generated if the set $X$ is coherent.

(ii) The ideal $I^\ast(X)$ is finitely generated if and only if the set $X$ is coherent.

**Remark 3.** (i) If $X$ is a closed coherent analytic subset of $\mathbb{R}^n$, then, in general, the ideal $I(X)$ is not finitely generated (cf. [3] for an idea of how to construct an example).
(ii) Theorem 2 applies, in particular, in the case where \( U = \mathbb{R}^n \) and \( X \) is an algebraic subset of \( \mathbb{R}^n \). We wish to point out that in such a case the ideal \( I(X) \) is not, in general, generated by polynomials. For example, the curve \( X \) in \( \mathbb{R}^2 \) given by the equation \( x_2^2 - x_1^2(x_1 - 1) = 0 \) is irreducible as an algebraic set and has an isolated point at the origin. It follows that the ideal \( I(X) \) cannot be generated by polynomials. Clearly, \( X \) is coherent as an analytic set.

There is a conjecture (cf. [4, p. 65]) that the ideal \( I(X) \) is always generated by so-called Nash functions if \( X \) is an algebraic subset of \( \mathbb{R}^n \).

Let us mention that if \( Y \) is a complex algebraic subset of \( \mathbb{C}^n \), then the ideal of holomorphic functions on \( \mathbb{C}^n \) vanishing on \( Y \) is generated by polynomials and, hence, is finitely generated [9].

**Proposition 4.** Let \( X \) be an algebraic subset of \( \mathbb{R}^n \). If the set of singular points of \( X \) is bounded, then the ideal \( I(X) \) is finitely generated.

We conclude this section by giving

**Example 5.** Let \( f(x_1, x_2, x_3) = x_3(x_1^2 + x_2^2) - x_1 \) and let \( X \) be the set of zeros of \( f \). One checks easily that the set of singular points of \( X \) is not bounded and \( X \) is not coherent as an analytic set. However, the ideal \( I(X) \) is generated by \( f \) [2, Proposition 5.2(1)]. By Theorem 2, the ideal \( I(X) \) is not finitely generated. (It is stated in [2, p. 85] that the ideal \( I(X) \) is not generated by \( f \) but the proof of this fact is incorrect.)

2. The proofs. The proofs are based on the results of Malgrange, Merrien and Tougeron.

**Proof of Proposition 1.** If the set \( X \) is coherent, then \( J_*(X) = J(X)\mathcal{E}|U \) [12, p. 127, Theorem 4.2]. Since each stalk of \( J(X) \) is finitely generated so is each stalk of \( J_*(X) \).

Now assume that each stalk of \( J_*(X) \) is finitely generated. Let \( \mathcal{F}_n \) be the \( \mathbb{R} \)-algebra of formal power series in \( n \) variables. Given a point \( x \) in \( \mathbb{R}^n \), denote by \( T_x : \mathcal{E}_x \to \mathcal{F}_n \) the homomorphism induced by the infinite Taylor expansion at \( x \). It follows from [7, p. 90, Theorem 3.5] that \( T_x(J_*(X)_x) = T_x(J(X)_x)\mathcal{F}_n \) for all \( x \) in \( U \). Fix a point \( x \) in \( U \) and let \( f_1, \ldots, f_k \) be analytic functions defined on a neighborhood \( U_x \) of \( x \) in \( U \) whose germs \( f_1x, \ldots, f_kx \) at \( x \) generate \( J(X)_x \). Then \( T_x(f_1), \ldots, T_x(f_k) \) generate \( T_x(J_*(X)_x) \). Clearly, \( J_*(X)_x \) is a closed ideal of \( \mathcal{E}_x \) (cf. [12, pp. 98–99] for a definition of a closed ideal in \( \mathcal{E}_x \)). By [5, p. 48, Proposition 1], the germs \( f_1x, \ldots, f_kx \) generate \( J_*(X)_x \). Since, by [1, Proposition 1.3], the sheaf \( J_*(X) \) is quasi-flasque, the germs \( f_1y, \ldots, f_ky \) generate \( J_*(X)_y \) for all \( y \) in \( U_x \) provided that \( U_x \) is a sufficiently small neighborhood of \( x \) [12, p. 115, Proposition 6.4]. Therefore \( T_y(f_1), \ldots, T_y(f_k) \) generate \( T_y(J(X)_y)\mathcal{F}_n \). By [12, p. 26, Proposition 8.2], the germs \( f_1y, \ldots, f_ky \) generate \( J(X)_y \) for all \( y \) in \( U_x \) and, hence, \( X \) is a coherent set.

**Proof of Theorem 2.** We claim that there exists a positive integer \( k \) such that the stalk \( J(X)_x \) can be generated by at most \( k \) elements of \( O_x \) for all \( x \) in \( U \). Indeed, define \( h : \mathbb{R}^n \to \mathbb{R}^n \) by \( h(x) = x/(1 + \|x\|^2)^{1/2} \) for \( x \) in \( \mathbb{R}^n \). Since the graph of \( h \) is semi-algebraic, the set \( Y = h(X) \) is also semi-algebraic [10].
The subsheaf $J(Y)$ of $\mathcal{O}$ over $\mathbb{R}^n$ is semifinite. Thus for each point $x$ in $\mathbb{R}^n$ there exist a neighborhood $U_x$ of $x$ in $\mathbb{R}^n$ and a positive integer $k_x$ such that $J(Y)_y$ can be generated by at most $k_x$ elements of $\mathcal{O}_y$ for all $y$ in $U_x$. Since $h$ diffeomorphically maps $\mathbb{R}^n$ onto the open unit ball in $\mathbb{R}^n$, the set $Y$ is bounded and the claim follows.

If $X$ is coherent, then the ideal $\Gamma(U, J(X))$ of global sections of $J(X)$ is finitely generated (cf. [6]). Since, by Theorem A of Cartan, $I(X) = \Gamma(U, J(X))$, the proof of (i) is completed.

Again, if $X$ is coherent, then $J_\ast(X) = J(X)\mathcal{E}|U$ [12, p. 127, Theorem 4.2]. It follows from [12, p. 214, Proposition 5.1] that the ideal $I_\ast(X) = \Gamma(U, J_\ast(X))$ is finitely generated.

Now assume that the ideal $I_\ast(X)$ is finitely generated. Clearly, $J_\ast(X)_x = I_\ast(X)_x$ for all $x$ in $U$. By Proposition 1, the set $X$ is coherent.

**PROOF OF PROPOSITION 3.** Let $J$ be the sheaf of ideals associated with $I(X)$, i.e., $J_x = I(X)_x$ for $x$ in $\mathbb{R}^n$. By [11, Theorem 2], the sheaf $J$ is coherent. Note that if $x$ is a nonsingular point in $X$, then the ideal $J_x$ is generated by polynomials vanishing on $X$. Since the set of singular points of $X$ is bounded, there exists a positive integer $k$ such that the ideal $J_x$ can be generated by at most $k$ elements of $\mathcal{O}_x$ for all $x$ in $\mathbb{R}^n$. It follows that the ideal $I(X) = \Gamma(X, J)$ is finitely generated.

**BIBLIOGRAPHY**


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