ON THE PARTIAL SUMS OF CONVEX FUNCTIONS OF ORDER 1/2

RAM SINGH

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ABSTRACT. Let \( f(z) = z + a_2 z^2 + \cdots \) be regular and univalently convex of order 1/2 in the unit disc \( U \) and let \( s_n(z, f) \) denote its nth partial sum. In the present note we determine the radius of convexity of \( s_n(z, f) \), depending on \( n \), and generalize and sharpen a result of Ruscheweyh concerning the partial sums of convex functions. We also prove that for every \( n \geq 1 \), \( \text{Re}(s_n(z, f)/z) > 1/2 \) in \( U \).

1. Let \( S \) denote the class of functions
\[
f(z) = z + \sum_{k=2}^{\infty} a_n z^n
\]
which are regular and univalent in the unit disc \( U = \{ z \mid |z| < 1 \} \). Denote by \( S_t \) and \( K \) the usual subclasses of \( S \) consisting of functions which map \( U \) onto starlike (with respect to the origin) and convex domains, respectively. Let \( S_t(1/2) \subset S_t \) be the class of functions which are starlike of order 1/2. Similarly, let \( K(1/2) \subset K \) be the family of functions which are convex of order 1/2. It is well known that \( K \subset S_t(1/2) \). Let
\[
s_n(z, f) = z + \sum_{k=2}^{n} a_k z^k
\]
denote the nth partial sum of \( f(z) \). Kobori [2] proved that if \( f \in K \), then for each \( n \geq 1 \), \( s_n(z, f) \) is starlike and univalent in \( |z| < 1/2 \). Using the theory of convolution, Ruscheweyh [5] in 1972 obtained the following result.

THEOREM A. Let \( r_n \) denote the positive root of
\[
\varphi_n(r) = 1 - (n + 1) r^n - n r^{n+1} \quad (n \in \mathbb{N}),
\]
and let \( f \in K \). Then \( s_n(z, f) \) is univalent for \( |z| < r_n \) and maps this disc onto a close-to-convex domain. For \( n \) even, \( r_n \) cannot be replaced by any larger number.

Robertson [3] in 1981 proved that if \( f \in K(1/2) \), then each \( s_n(z, f) \) is close-to-convex with respect to \( f \) and hence univalent in \( U \).

Using the fact that the class \( K \) is closed with respect to Hadamard convolution, we can readily compute the radius of convexity of \( s_n(z, f) \), \( f \in K \), depending on \( n \). In the present paper we determine the radius of convexity of \( s_n(z, f) \), \( f \in K(1/2) \), in terms of \( n \). As a corollary to our result, we shall show that one can weaken the hypothesis of Ruscheweyh's theorem and still make a stronger assertion. We shall also prove that if \( f \in K(1/2) \), then \( \text{Re}(s_n(z, f)/z) > 1/2 \) (\( z \in U \)) for each \( n \in \mathbb{N} \).
2. We shall need the following definitions and results.

Let \( f \) and \( g \) be analytic in \(|z| < R\) and \( f(0) = g(0)\). In addition, suppose that \( g \) is univalent in \(|z| < R\). Then we say that \( f \) is subordinate to \( g \) in \(|z| < R\), in symbols, \( f \prec g(|z| < R) \), if \( f(|z| < R) \subset g(|z| < R) \).

**Lemma 1.** If \( f \) is analytic in \( U \) and \( f(0) = f'(0) - 1 = 0 \), then \( f \in S_t(1/2) \) if and only if

\[
\text{Re} \left( \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right) > \frac{1}{2}
\]

for \(|z_1| < 1, |z_2| < 1\).

**Lemma 2.** For \( 0 \leq \theta \leq \pi \),

\[
\frac{1}{2} + \sum_{k=1}^{n} \frac{1}{k + 1} \cos k\theta \geq 0.
\]

Lemma 1 is due to Ruscheweyh and Sheil-Small [6] and Lemma 2 is due to Rogosinski and Szegö [4].

3. We now prove the following:

**Theorem 1.** Let \( f \in K(1/2) \) and let \( r_n \) be defined as in Theorem A. Then \( s_n(z, f) \) maps the disc \(|z| < r_n\) onto a convex domain. For even \( n \), the number \( r_n \) cannot be replaced by any larger one.

**Proof.** Since \( f \in K(1/2) \), \( g(z) = zf'(z) \in S_t(1/2) \) and, therefore, in view of Lemma 1, it follows that for all \( z \) and \( \xi \) in \( U \),

\[
\text{Re} \left( \frac{g(z) - g(\xi)}{z - \xi} \right) > \frac{1}{2}
\]

Treating \( \xi \) as a constant, this leads to

\[
\text{Re} \left[ 1 + \frac{1}{\xi} \left( 1 - \frac{s_1(\xi, g)}{g(\xi)} \right) z + \frac{1}{\xi^2} \left( 1 - \frac{s_2(\xi, g)}{g(\xi)} \right) z^2 + \cdots + \frac{1}{\xi^n} \left( 1 - \frac{s_n(\xi, g)}{g(\xi)} \right) z^n + \cdots \right] > \frac{1}{2}
\]

(z in \( U \)), and, consequently, for all \( n \geq 1 \), we get

\[
\left| \frac{1}{\xi^n} \left( 1 - \frac{s_n(\xi, g)}{g(\xi)} \right) \right| \leq 1 \quad [1, p. 41].
\]

Replacing \( \xi \) by \( z \) and writing \( g \) in terms of \( f \), the above inequality provides

\[
1 - \frac{s_n'(z, f)}{f''(z)} = z^{n-1} \varphi(z),
\]

where \( \varphi \) is analytic, \( \varphi(0) = 0 \) and \(|\varphi(z)| \leq |z| \) in \( U \). From (1), we get

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \text{Re} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right) - z^{n-1} \frac{((n-1)\varphi(z) + z\varphi'(z))}{1 - z^{n-1}\varphi(z)} \right]
\]

\[
\geq \frac{1}{1 + r} - \frac{r^{n-1}}{(1 - r^2)} \left[ \frac{(n-1)(1 - r^2)t + r(1 - t^2)}{(1 - r^{n-1}t)} \right],
\]
where \( r = |z|, \ t = |\varphi(z)| \) and use has been made of the well-known inequalities:

\[
|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2},
\]

\[
\text{Re} \left( 1 + \frac{z f''(z)}{f'(z)} \right) \geq \frac{1}{1 + |z|} \quad (f \in K(1/2)).
\]

Letting

\[
\psi(t) = \frac{(n - 1)(1 - t^2)t + r(1 - t^2)}{1 - r^{n-1}t},
\]

we readily see that

\[
N(\psi'(t)) = \text{the numerator of } \psi'(t)
\]

\[
= (n - 1)(1 - r^2) + r^n - 2rt + r^n t^2,
\]

and that \((\partial / \partial t)N(\psi'(t)) \leq 0\).

\[
\therefore \min N(\psi'(t)) = N(\psi'(r)) = (n - 1) - (n + 1)r^2 + r^n + 2
\]

\[
= (r - 1)^2[r^n + 2r^{n-1} + 4r^{n-2} + \cdots + 2(n - 1)r + (n - 1)] > 0.
\]

Thus we conclude that \( \psi'(t) > 0 \) for all admissible values of \( t \) and as such

\[
\text{Re} \left( 1 + \frac{z s_n''(z,f)}{s'(z,f)} \right) \geq \frac{1}{1 + r} - \frac{n r^n}{1 - r^n}
\]

\[
= \frac{1 - (n + 1)r^n - nr^{n+1}}{(1 + r)(1 - r^n)}.
\]

\( s_n(z,f) \) will, therefore, map the disc \(|z| < r_n\) onto a convex domain, where \( 0 < r_n < 1 \) is the unique positive root of the polynomial \( \varphi_n(r) \), defined in Theorem A. This proves the first assertion of our theorem. To prove the second assertion, let \( n = 2m \) be any even positive integer, and consider the function \( f_0(z) = -\log(1 - z) \), which belongs to \( K(1/2) \). Then

\[
1 + \frac{s_{2m}'(z, f_0)}{s_{2m}'(z, f_0)} = \frac{1 + 2z + 3z^2 + \cdots + 2mz^{2m-1}}{1 + z + z^2 + \cdots + z^{2m-2}}
\]

\[
= \frac{1 - (2m + 1)z^{2m} + 2mz^{2m+1}}{(1 - z)(1 - z^{2m})},
\]

and, therefore,\[
\left[ 1 + \frac{z s_{2m}'(z, f_0)}{s'(z, f_0)} \right]_{z = -r} = \frac{1 - (n + 1)r^n - nr^{n+1}}{(1 + r)(1 - r^n)}
\]

\[
= 0, \quad \text{when } r = r_n.
\]

The proof of Theorem 1, is, therefore, complete.

**Corollary 1.** If \( f \in S_t(1/2) \), then \( s_n(z,f) \) maps the disc \(|z| < r_n\) onto a domain which is starlike with respect to the origin. For even \( n \), the number \( r_n \) cannot be replaced by any larger one.

Since \( K \subset S_t(1/2) \), Corollary 1 shows that one can weaken the hypothesis of Ruscheweyh's theorem \((K \text{ may be replaced by } S_t(1/2))\) and still make a stronger assertion \(|z| < r_n \text{ is mapped onto a starlike domain}\).
As Ruscheweyh [5] has shown, the number $r_n$ satisfies

$$(2n)^{-1/n} \leq r_n \leq n^{-1/(n-1)}$$

and

$$r_n = 1 - \frac{\log(2n)}{n} + \frac{\log(n) \log(4en)}{2n^2} + o\left(\frac{\log(n)}{n^2}\right).$$

**Theorem 2.** Let $f \in K(1/2)$. Then for each $n \geq 1$,

$$\text{Re} \left( \frac{s_n(z,f)}{z} \right) > \frac{1}{2} \quad (z \in U).$$

The constant $1/2$ cannot be replaced by any larger one.

**Proof.** Since $f \in K(1/2)$, the function $zf'(z) \in S_t(1/2)$ and, hence,

$$\text{Re} \ f'(z) > \frac{1}{2} \quad (z \in U).$$

Now we can write

$$\frac{s_n(z,f)}{z} = f'(z) \ast \left[ 1 + \sum_{k=2}^{n} \frac{z^{k-1}}{k} \right]$$

$$= f'(z) \ast \left[ 1 + \sum_{k=1}^{n-1} \frac{z^k}{k+1} \right],$$

where $\ast$ denotes Hadamard convolution.

Putting $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$, and making use of the minimum principle for harmonic functions along with Lemma 2, we obtain

$$\text{Re} \left[ 1 + \sum_{k=1}^{n-1} \frac{r^k \cos k\theta}{k+1} \right] = 1 + \sum_{k=1}^{n-1} \frac{r^k \cos k\theta}{k+1} \geq \frac{1}{2}.$$  

(4)

From (2), (4) and (3) we deduce that $\text{Re}(s_n(z,f)/z) > 1/2$ in $U$. Since for the function $f_0(z) = -\log(1 - z)$ ($\in K(1/2)$), we have $(s_2(z,f)/z) = 1 + (z/2)$, the sharpness of the number $1/2$ is obvious.

**Remark.** It is clear from the proof of Theorem 2 that its assertion holds for the wider class of functions $f$ which are regular in and satisfy the conditions $\text{Re} f'(z) > 1/2$, $z \in U$.

It is well known that if $f \in K$, then $z/2 = s_1(z,f)/2 < f(z)$ in $U$. Since each $s_n(z,f)$, $f \in K(1/2)$, is univalent in $U$, it is natural to ask for the largest number $\lambda_n$, $0 < \lambda_n < 1$, such that $\lambda_n s_n(z,f) < s_{n+1}(z,f)$ in $U$. The following theorem, which we state without proof, provides a lower bound for $\lambda_n$.

**Theorem 3.** If $f \in K(1/2)$, then

(i) $z/2 = \frac{1}{2} s_1(z,f) < s_2(z,f)$, $(z \in U)$,

and for $n \geq 2$,

(ii) $((n-1)/(n+1))s_n(z,f) < s_{n+1}(z,f)$, $(z \in U)$.

The number $1/2$ in (i) cannot be replaced by any larger one.
REFERENCES


DEPARTMENT OF MATHEMATICS, PUNJABI UNIVERSITY, PATIALA-147002, INDIA