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ON THE PARTIAL SUMS OF CONVEX FUNCTIONS OF ORDER 1/2

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ABSTRACT. Let $f(z) = z + a_2z^2 + \cdots$ be regular and univalently convex of order 1/2 in the unit disc U and let $s_n(z, f)$ denote its nth partial sum. In the present note we determine the radius of convexity of $s_n(z, f)$, depending on n, and generalize and sharpen a result of Ruscheweyh concerning the partial sums of convex functions. We also prove that for every $n \geq 1$, $\mathrm{Re}(s_n(z, f)/z) > 1/2$ in U.

1. Let S denote the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_n z^n$$

which are regular and univalent in the unit disc $U = \{z \mid |z| < 1\}$. Denote by S_t and K the usual subclasses of S consisting of functions which map U onto starlike (with respect to the origin) and convex domains, respectively. Let $S_t(1/2) \subset S_t$ be the class of functions which are starlike of order 1/2. Similarly, let $K(1/2) \subset K$ be the family of functions which are convex of order 1/2. It is well known that $K \subset S_t(1/2)$. Let

$$s_n(z, f) = z + \sum_{k=2}^{n} a_k z^k$$

denote the nth partial sum of $f(z)$. Kobori [**2**] proved that if $f \in K$, then for each $n \geq 1$, $s_n(z, f)$ is starlike and univalent in $|z| < 1/2$. Using the theory of convolution, Ruscheweyh [**5**] in 1972 obtained the following result.

THEOREM A. *Let r_n denote the positive root of*

$$\varphi_n(r) = 1 - (n+1)r^n - nr^{n+1} \quad (n \in \mathbf{N}),$$

and let $f \in K$. Then $s_n(z, f)$ is univalent for $|z| < r_n$ and maps this disc onto a close-to-convex domain. For n even, r_n cannot be replaced by any larger number.

Robertson [**3**] in 1981 proved that if $f \in K(1/2)$, then each $s_n(z, f)$ is close-to-convex with respect to f and hence univalent in U.

Using the fact that the class K is closed with respect to Hadamard convolution, we can readily compute the radius of convexity of $s_n(z, f)$, $f \in K$, depending on n. In the present paper we determine the radius of convexity of $s_n(x, f)$, $f \in K(1/2)$, in terms of n. As a corollary to our result, we shall show that one can weaken the hypothesis of Ruscheweyh's theorem and still make a stronger assertion. We shall also prove that if $f \in K(1/2)$, then $\mathrm{Re}(s_n(z, f)/z) > 1/2$ $(z \in U)$ for each $n \in \mathbf{N}$.

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2. We shall need the following defintions and results.

Let f and g be analytic in $|z| < R$ and $f(0) = g(0)$. In addition, suppose that g is univalent in $|z| < R$. Then we say that f is subordinate to g in $|z| < R$, in symbols, $f \prec g(|z| < R)$, if $f(|z| < R) \subset g(|z| < R)$.

LEMMA 1. *If f is analytic in U and $f(0) = f'(0) - 1 = 0$, then $f \in S_t(1/2)$ if and only if*

$$\operatorname{Re}\left(\frac{f(z_1) - f(z_2)}{z_1 - z_2}\,\frac{z_2}{f(z_2)}\right) > \frac{1}{2}$$

for $|z_1| < 1$, $|z_2| < 1$.

LEMMA 2. *For $0 \le \theta \le \pi$,*

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+1} \ge 0.$$

Lemma 1 is due to Ruscheweyh and Sheil-Small [**6**] and Lemma 2 is due to Rogosinski and Szegö [**4**].

3. We now prove the following:

THEOREM 1. *Let $f \in K(1/2)$ and let r_n be defined as in Theorem* A. *Then $s_n(z, f)$ maps the disc $|z| < r_n$ onto a convex domain. For even n, the number r_n cannot be replaced by any larger one.*

PROOF. Since $f \in K(1/2)$, $g(z) = zf'(z) \in S_t(1/2)$ and, therefore, in view of Lemma 1, it follows that for all z and ς in U,

$$\operatorname{Re}\left(\frac{g(z) - g(\varsigma)}{z - \varsigma}\,\frac{\varsigma}{g(\varsigma)}\right) > \frac{1}{2}.$$

Treating ς as a constant, this leads to

$$\operatorname{Re}\left[1 + \frac{1}{\varsigma}\left(1 - \frac{s_1(\varsigma, g)}{g(\varsigma)}\right) z + \frac{1}{\varsigma^2}\left(1 - \frac{s_2(\varsigma, g)}{g(\varsigma)}\right) z^2 + \cdots + \frac{1}{\varsigma^n}\left(1 - \frac{s_n(\varsigma, g)}{g(\varsigma)}\right) z^n + \cdots\right] > \frac{1}{2} \qquad (z \in U),$$

and, consequently, for all $n \ge 1$, we get

$$\left|\frac{1}{\varsigma^n}\left(1 - \frac{s_n(\varsigma, g)}{g(\varsigma)}\right)\right| \le 1 \quad [\mathbf{1}, \text{p. } 41].$$

Replacing ς by z and writing g in terms of f, the above inequality provides

$$1 - \frac{s_n'(z, f)}{f'(z)} = z^{n-1}\varphi(z), \tag{1}$$

where φ is analytic, $\varphi(0) = 0$ and $|\varphi(z)| \le |z|$ in U. From (1), we get

$$\begin{aligned}\operatorname{Re}\left(1 + \frac{zs''(z, f)}{s'(z, f)}\right) &= \operatorname{Re}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) - z^{n-1}\frac{((n-1)\varphi(z) + z\varphi'(z))}{1 - z^{n-1}\varphi(z)}\right] \\ &\ge \frac{1}{1+r} - \frac{r^{n-1}}{(1-r^2)}\left[\frac{(n-1)(1-r^2)t + r(1-t^2)}{(1-r^{n-1}t)}\right],\end{aligned}$$

where $r = |z|$, $t = |\varphi(z)|$ and use has been made of the well-known inequalities:

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2},$$

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \frac{1}{1 + |z|} \qquad (f \in K(1/2)).$$

Letting

$$\psi(t) = \frac{(n-1)(1-r^2)t + r(1-t^2)}{1 - r^{n-1}t},$$

we readily see that

$$\begin{aligned} N(\psi'(t)) &= \text{ the numerator of } \psi'(t) \\ &= (n-1)(1-r^2) + r^n - 2rt + r^n t^2, \end{aligned}$$

and that $(\partial/\partial t)N(\psi'(t)) \leq 0$.

$$\begin{aligned} \therefore \min N(\psi'(t)) &= N(\psi'(r)) = (n-1) - (n+1)r^2 + r^n + r^n + 2 \\ &= (r-1)^2[r^n + 2r^{n-1} + 4r^{n-2} + \cdots + 2(n-1)r + (n-1)] > 0. \end{aligned}$$

Thus we conclude that $\psi'(t) > 0$ for all admissible values of t and as such

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{zs_n''(z,f)}{s'(z,f)}\right) &\geq \frac{1}{1+r} - \frac{nr^n}{1-r^n} \\ &= \frac{1 - (n+1)r^n - nr^{n+1}}{(1+r)(1-r^n)}. \end{aligned}$$

$s_n(z,f)$ will, therefore, map the disc $|z| < r_n$ onto a convex domain, where $0 < r_n < 1$ is the unique positive root of the polynomial $\varphi_n(r)$, defined in Theorem A. This proves the first assertion of our theorem. To prove the second assertion, let $n = 2m$ be any even positive integer, and consider the function $f_0(z) = -\log(1-z)$, which belongs to $K(1/2)$. Then

$$\begin{aligned} 1 + \frac{s_{2m}''(z,f_0)}{s_{2m}'(z,f_0)} &= \frac{1 + 2z + 3z^2 + \cdots + 2mz^{2m-1}}{1 + z + z^2 + \cdots + z^{2m-1}} \\ &= \frac{1 - (2m+1)z^{2m} + 2mz^{2m+1}}{(1-z)(1-z^{2m})}, \end{aligned}$$

and, therefore,

$$\begin{aligned} \left[1 + \frac{zs_n''(z,f_0)}{s'(z,f_0)}\right]_{z=-r} &= \frac{1 - (n+1)r^n - nr^{n+1}}{(1+r)(1-r^n)} \\ &= 0, \quad \text{when } r = r_n. \end{aligned}$$

The proof of Theorem 1, is, therefore, complete.

COROLLARY 1. *If $f \in S_t(1/2)$, then $s_n(z,f)$ maps the disc $|z| < r_n$ onto a domain which is starlike with respect to the origin. For even n, the number r_n cannot be replaced by any larger one.*

Since $K \subset S_t(1/2)$, Corollary 1 shows that one can weaken the hypothesis of Ruscheweyh's theorem (K may be replaced by $S_t(1/2)$) and still make a stronger assertion ($|z| < r_n$ is mapped onto a starlike domain).

As Ruscheweyh [5] has shown, the number r_n satisfies

$$(2n)^{-1/n} \le r_n \le n^{-1(n-1)}$$

and

$$r_n = 1 - \frac{\log(2n)}{n} + \frac{\log(n)\log(4en)}{2n^2} + o\left(\frac{\log(n)}{n^2}\right).$$

THEOREM 2. *Let $f \in K(1/2)$. Then for each $n \ge 1$,*

$$\operatorname{Re}\frac{s_n(z,f)}{z} > \frac{1}{2} \qquad (z \in U).$$

The constant 1/2 cannot be replaced by any larger one.

PROOF. Since $f \in K(1/2)$, the function $zf'(z) \in S_t(1/2)$ and, hence,

$$\operatorname{Re} f'(z) > \frac{1}{2} \qquad (z \in U). \tag{2}$$

Now we can write

$$\begin{aligned} \frac{s_n(z,f)}{z} &= f'(z) * \left[1 + \sum_{k=2}^{n} \frac{z^{k-1}}{k}\right] \\ &= f'(z) * \left[1 + \sum_{k=1}^{n-1} \frac{z^k}{k+1}\right], \end{aligned} \tag{3}$$

where $*$ denotes Hadamard convolution.

Putting $z = re^{i\theta}$, $0 \le r < 1$, $0 \le \theta < 2\pi$, and making use of the minimum principle for harmonic functions along with Lemma 2, we obtain

$$\begin{aligned} \operatorname{Re}\left[1 + \sum_{k=1}^{n-1} \frac{z^k}{k+1}\right] &= \left[1 + \sum_{k=1}^{n-1} \frac{r^k \cos k\theta}{k+1}\right] \qquad (0 \le \theta \le \pi) \\ &> 1 + \sum_{k=1}^{n-1} \frac{\cos k\theta}{k+1} \ge \frac{1}{2}. \end{aligned} \tag{4}$$

From (2), (4) and (3) we deduce that $\operatorname{Re}(s_n(z,f)/z) > 1/2$ in U. Since for the function $f_0(z) = -\log(1-z)$ $(\in K(1/2))$, we have $(s_2(z,f)/z) = 1 + (z/2)$, the sharpness of the number 1/2 is obvious.

REMARK. It is clear from the proof of Theorem 2 that its assertion holds for the wider class of functions f which are regular in and satisfy the conditions $\operatorname{Re} f'(z) > 1/2$, $z \in U$.

It is well known that if $f \in K$, then $z/2 = s_1(z,f)/2 \prec f(z)$ in U. Since each $s_n(z,f)$, $f \in K(1/2)$, is univalent in U, it is natural to ask for the largest number λ_n, $0 < \lambda_n < 1$, such that $\lambda_n s_n(z,f) \prec s_{n+1}(z,f)$ in U. The following theorem, which we state without proof, provides a lower bound for λ_n.

THEOREM 3. *If $f \in K(1/2)$, then*

(i) $z/2 = \frac{1}{2}s_1(z,f) \prec s_2(z,f)$, $(z \in U)$,

and for $n \ge 2$,

(ii) $((n-1)/(n+1))s_n(z,f) \prec s_{n+1}(z,f)$, $(z \in U)$.

The number 1/2 in (i) *cannot be replaced by any larger one.*

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