REMOVABLE SINGULARITIES IN THE NEVANLINNA CLASS
AND IN THE HARDY CLASSES

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ABSTRACT. We show that certain sets in $\mathbb{C}^n$, $n \geq 2$, which we call $n$-small, are removable singularities for holomorphic functions in the Nevanlinna class. Since our class of sets includes polar sets (in $\mathbb{R}^{2n}$) our result includes the previous removable singularity results for the Nevanlinna class. We give also a related result for a subclass of the Hardy class.

1. Throughout this paper, $G$ is an open set in $\mathbb{C}^n$, $n \geq 1$, and $E$ is closed in $G$. Parreau [12, Théorème 20, p. 182] gave essentially the following result: If $n = 1$, $E$ is polar, and $f$ is a holomorphic function in $G \setminus E$ such that $\log^+ |f|$ has a harmonic majorant in $G \setminus E$, then $f$ has a meromorphic extension to $G$. In answering a question of Cima and Graham [2, Remarks 7.4, p. 255], Parreau’s theorem was in [7, Theorem 3.4, p. 477] extended to the case $n \geq 2$ and $E$ polar in $\mathbb{R}^{2n}$.

The purpose of this note is to give a result, Theorem 1 below, which contains the above result as a special case. In Theorem 1 the exceptional set $E$ is allowed to be slightly larger, that is $n$-small (for the definition see [14, Definition 2.2, p. 101] or §3 below). However, we must then replace the condition that $\log^+ |f|$ has a harmonic majorant in $G \setminus E$ by the condition that the (Riesz) measure $\Delta \log^+ |f|$ has locally finite mass near the exceptional set $E$. In addition, we give in §6 a slight generalization to [8, Corollary 3.6, p. 301].

2. Let $u$ be a subharmonic function in $G \setminus E$. One says that the (Riesz) measure $\Delta u$ has locally finite mass near $E$, if $\Delta u(D \setminus E) < \infty$ for each open set $D \subseteq G$ (relatively compact in $G$). From [1, p. 283] (see also [7, Lemma 3.3, p. 476]), it follows that if $\Delta u$ has locally finite mass near $E$, then $u$ has locally a harmonic majorant near $E$ (that is, for each open set $D \subseteq G$ there is a harmonic function $h$ in $D \setminus E$ such that $u \leq h$ in $D \setminus E$). The converse holds in the important case when $E$ is polar in $\mathbb{R}^{2n}$ (see [1, p. 283] or the proof of Corollary 1 below), but not in general. To get a simple example, set $E' = \{z = x + iy \in \mathbb{C} | |x| = |y|\}$, $u(z) = -\log(\max\{|x|, |y|\})$ and $h(z) = \log \sqrt{2} - \log |z|$. Then in $\mathbb{C} \setminus E'$ the subharmonic function $u$ has the harmonic majorant $h$. However, one sees easily that $\Delta u$ does not have locally finite mass near $E'$.

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546
3. For $F \subset C$, set $C^1(F) = \text{cap}^* F$, where $\text{cap}^*$ is the outer logarithmic capacity (for the definition of $\text{cap}^*$, see [5, pp. 210, 273]). If $F \subset C^n$, $n \geq 2$, then set

$$C^n(F) = \max_{1 \leq j \leq n} H_2\{z_j \in C|C^{n-1}\{z_j, Z_j\} \in F\} > 0.$$

Here, and in the sequel, $z = (z_1, \ldots, z_j, \ldots, z_n) = (z_j, Z_j)$, $1 \leq j \leq n$, and $H_\alpha$ is the $\alpha$-dimensional Hausdorff outer measure. We say that $F \subset C^n$ is $n$-small, if $C^n(F) = 0$.

**Lemma 1** [7, Proposition 2.3, p. 472]. An $F_\alpha$-set $F \subset C^n$, $n \geq 2$, is $n$-small if and only if for each $j$, $1 \leq j \leq n$, $H_{2n-2}(F_j) = 0$, where

$$F_j = \{z_j \in C^{n-1}|\text{cap}^*\{z_j \in C|(z_j, Z_j) \in F\} > 0\}.$$

With the aid of this lemma it follows from a result of Mattila [10, Corollary 3.3, p. 263] (see also [15, Lemma 6, p. 115]) that polar sets in $R^{2n}$ are $n$-small.

If the Hausdorff measure $H_2$ in (A) is replaced by the outer logarithmic capacity $\text{cap}^*$, a set function which is sometimes denoted by $g_n$ is obtained. Sets $E$ for which $g_n(E) = 0$ have been used as exceptional sets at least in [1, 16, 17, and 14]. If $F \subset C^n$ and $g_n(F) = 0$, then clearly $F$ is $n$-small. On the other hand, there are sets $F \subset C^n$ with $g_n(F) = 0$ which are not even polar. In fact, with the help of [3, Theorem 2, p. 118] one can construct a compact set $F \subset C^2$ of Hausdorff dimension 4 such that $g_2(F) = 0$ and such that for each $z \in C$ the sections $F \cap (C \times \{z\})$ and $F \cap (\{z\} \times C)$ contain at most one point.

4. Next we give the lemmas we need in the sequel.

**Lemma 2** [18, Theorem 1.2, p. 16]. Let $G'$ be a domain of $C^{n-1}$, $n \geq 2$, and $F \subset G'$. Suppose $j \in \mathbb{N}$, $1 \leq j \leq n$, $r'_j, r''_j \in \mathbb{R}$, $0 < r'_j < r''_j$, and $z_j^0 \in C$. Let $f$ be a holomorphic function in the open set

$$V(G', z_j^0, r'_j, r''_j) = \{z = (z_j, Z_j) \in C^n|z_j \in A(z_j^0, r'_j, r''_j), Z_j \in G'\}$$

such that for each $Z_j \in F$ the holomorphic function $f_{Z_j}: A(z_j^0, r'_j, r''_j) \to C$,

$$f_{Z_j}(z_j) = f(z_j, Z_j) = f(z),$$

has a meromorphic extension to $B^2(z_j^0, r'_j)$. If $F$ is not contained in a countable union of analytic subvarieties in $G'$ of codimension $\geq 1$, then $f$ has a meromorphic extension to the open set

$$V(G', z_j^0, r'_j) = \{z = (z_j, Z_j) \in C^n|z_j \in B^2(z_j^0, r'_j), Z_j \in G'\}.$$

Here, $B^2(z_0, r)$ is the disc in $C$ with center $z_0$ and radius $r$, and $A(z_0, r_1, r_2)$ is the annulus $B^2(z_0, r_2) \setminus B^2(z_0, r_1)$. Following Siu [18, p. 17] we define the radius of meromorphy as follows (note that also other definitions are used, see for example [4, p. 578]). Let $f$ be a holomorphic function in the set $V(G', z_j^0, r'_j)$ where $G', j$, and $z_j^0$ are as above, and $r_j \in \mathbb{R}_+$. For each $Z_j \in G'$ the radius of meromorphy $\rho_j(Z_j)$ is the supremum of all $\rho > 0$ such that $f$ has a meromorphic extension to a neighborhood of the set

$$V(Z_j, z_j^0, \rho) = \{z = (z_j, Z_j) \in C^n|z_j \in B^2(z_j^0, \rho)\}.$$
With this notation, we have

**Lemma 3** [18, Proposition 1.4 and Remark 1.5, pp. 17-18]. The function \( v_j : G' \to [-\infty, \infty), v_j(Z_j) = -\log p_j(Z_j) \), is subharmonic.

5. If \( f \) is a holomorphic function in \( G \setminus E \) and has a meromorphic extension \( f^* \) to \( G \), then the measure \( \Delta \log^+ |f| \) has locally finite mass near \( E \). This follows from the fact that each point \( z_0 \in G \) has a neighborhood \( U \) in \( G \) such that \( \log^+ |f^*| \) has a pluriharmonic majorant in \( U \setminus N(f^*) \). Here, \( N(f^*) \) is the nonsmooth set of \( f^* \); see, for example, [19, pp. 184-185]. As for the converse, we have

**Theorem 1.** Let \( E \) be \( n \)-small. Let \( f \) be a holomorphic function in \( G \setminus E \). If \( \Delta \log^+ |f| \) has locally finite mass near \( E \), then \( f \) has a meromorphic extension to \( G \).

**Proof.** If \( n = 1 \), then \( E \) is polar. Thus \( \log^+ |f| \) has locally a harmonic majorant near \( E \), and the theorem follows from [12, Théorème 20, p. 182].

Suppose then \( n \geq 2 \). It is sufficient to show that each point \( z^* \in E \) has a neighborhood \( V \) such that \( f|V \setminus E \) has a meromorphic extension to \( V \). As for the notation not explained in the sequel, see above.

Take \( z^* \in E \) arbitrarily and choose \( r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+ \) such that

\[
U = D^n(z^*, 2r) = B^2(z_1^*, 2r_1) \times \cdots \times B^2(z_n^*, 2r_n) \subset G.
\]

For shortness, we write \( u = \log^+ |f| \) \( \mid U \setminus E \). Since \( \Delta u(U \setminus E) < \infty \), one sees, proceeding as Cegrell [1, proof of Theorem, pp. 284-285], using a nondecreasing sequence of nonnegative testfunctions in \( U \setminus E \) tending to \( 1 \), using the \( n \)-subharmonicity of \( u \), Fubini’s theorem, and the Monotone convergence theorem, that for each \( j, 1 \leq j \leq n \), there is a set \( B_j \subset \mathbb{C}^n \) such that \( H_{2n-2}(B_j) = 0 \) and that for each \( Z_j \in U(z_j^*) \setminus B_j \) the measure \( \Delta u_{Z_j} \) has locally finite mass near the section \( (E \cap U)(Z_j) \). For a detailed discussion of this, see [7, proof of Theorem 3.4, pp. 477-478]. Since \( E \) is \( n \)-small, by Lemma 1 we may suppose that the section \( (E \cap U)(Z_j) \) is polar in \( \mathbb{C} \) for each \( Z_j \in U(z_j^*) \setminus B_j \). Above we have used the notation

\[
U(z_j^*) = \{ Z_j \in \mathbb{C}^n \mid z = (z_j^*, Z_j) \in U \},
\]

\[
(E \cap U)(Z_j) = \{ z_j \in \mathbb{C} \mid z = (z_j, Z_j) \in U \cap E \}
\]

for the sections of \( U \) in \( \mathbb{C}^n \) and of \( E \cap U \) in \( \mathbb{C} \), respectively. Take \( Z_j' \in U(z_j^*) \setminus B_j \) arbitrarily. Since the section \( (E \cap U)(Z_j') \) is then polar in \( \mathbb{C} \) and \( E \) is closed in \( G \), there are \( r_j', r_j'' \), \( 0 < r_j' < r_j'' < 2r_j \) and a (connected) neighborhood \( W \subset U(z_j^*) \) of \( Z_j' \) such that \( V(W, z_j^*, r_j', r_j'') \subset G \setminus E \). Since for each \( Z_j \in W \setminus B_j \) the measure \( \Delta u_{Z_j} \) has locally finite mass near the polar set \( (E \cap U)(Z_j) \), as shown above, we see as in the case \( n = 1 \) treated above, that the holomorphic functions \( f_{Z_j} B^2(z_j, 2r_j) \setminus (E \cap U)(Z_j), Z_j \in W \setminus B_j \), have meromorphic extensions to \( B^2(z_j^*, 2r_j) \). Invoking this and the facts that \( f \) is holomorphic in \( V(W, z_j^*, r_j', r_j'') \) and \( H_{2n-2}(B_j) = 0 \), we see by Lemma 2 that \( f \) has a meromorphic extension to the set \( V(W, z_j^*, r_j') \). Since \( r_j'' \) can be chosen arbitrarily near to \( 2r_j' \), we have shown the following: For each \( j, 1 \leq j \leq n \), and \( Z_j \in U(z_j^*) \setminus B_j \), the function \( f \) has a meromorphic extension to a neighborhood of \( E \). We refer to this result as condition (B).

Take \( z_0 \in D^n(z^*, r) \) and \( r^0 = (r_0, \ldots, r_n) \in \mathbb{R}^n_+ \) such that \( V_0 = D^n(z_0, r^0) \subset D^n(z^*, r) \setminus E \). By (B) we see that \( \rho_1(Z_1) \geq r_1 \) for each \( Z_1 \in V_0(z_0) \setminus B_1 \). Thus
by Lemma 3 and, for example, by [6, Proposition 2b'), p. 10], \( \rho_1(Z_1) \geq r_1 \) for all \( Z_1 \in V_0(z_0^0) \), the section of \( V_0 \). Thus \( f \) has a meromorphic extension \( f_1 \) to \( V_1 = D^n(z_0, r^1) \), where \( r^1 = (r_1, r_0^2, \ldots, r_0^n) \). Observe that \( z_j^* \in B^2(z_j^0, r_j) \). For the induction step suppose that \( 1 \leq k < n \) and \( r^k = (r_1, \ldots, r_k, r_{k+1}^0, \ldots, r_n^0) \in \mathbb{R}^n_+ \), and \( f_k \) is a holomorphic extension of \( f \) to \( V_k = D^n(z_0, r^k) \). Observe that \( z_j^* \in B^2(z_j^0, r_j) \), \( j = 1, \ldots, k \). As above, using (B) we see that \( \rho_{k+1}(Z_{k+1}) \geq r_{k+1} \) for all \( Z_{k+1} \in V_k(z_{k+1}^0) \). Thus \( f_k \) and hence also \( f \) has a meromorphic extension \( f_{k+1} \) to \( V_{k+1} = D^n(z_0, r^{k+1}) \), where \( r^{k+1} = (r_1, \ldots, r_{k+1}, r_{k+2}^0, \ldots, r_n^0) \). Moreover, \( z_j^* \in B^2(z_j^0, r_j) \), \( j = 1, \ldots, k+1 \), and \( V_{k+1} \subset U \). Thus the induction step is complete. Since \( V_n = D^n(z_0, r^n) \) is a neighborhood of \( z^* \), the proof is finished.

REMARK. The proof can also be based on [4, Theorem 2.9, p. 578] instead of Lemmas 2 and 3.

COROLLARY 1 [7, Theorem 3.4, p. 477]. Let \( E \) be polar (respectively \( n \)-small). Let \( f \) be a holomorphic function in \( G \setminus E \). If \( \log^+ |f| \) has a harmonic majorant (respectively \( n \)-superharmonic majorant which is \( \not\equiv \infty \) on each component of \( G \setminus E \)), then \( f \) has a meromorphic extension to \( G \).

PROOF. By [6, Theorem 2, p. 25] (respectively [14, Theorem 4.1, p. 105]) the harmonic (respectively \( n \)-superharmonic) majorant \( h \) to \( \log^+ |f| \) has a superharmonic extension \( h^* \) to \( G \). Similarly, the subharmonic (respectively \( n \)-subharmonic) function \( \log^+ |f| - h \) has a subharmonic extension \( h^*_1 \) to \( G \). Since \( \log^+ |f| = h^* + h^*_1 \) in \( G \setminus E \) and \( -\Delta h^* \) and \( \Delta h^*_1 \) are measures in \( G \), the corollary follows.

6. Computing the Laplacian, using Kametani’s extension result [11, Theorem 2, p. 10, and Remark, p. 11] and proceeding as in the proof of Theorem 1 above, the following result will be proved in [8, Corollary 3.6, p. 301]: Let \( H_{2n-1}(E) = 0 \). Let \( f \) be a holomorphic function in \( G \setminus E \). If for some \( p > 0 \) the measure \( \Delta |f|^p \) has locally finite mass near \( E \), then \( f \) has a holomorphic extension to \( G \). The proof works also for measures \( \Delta (\log^+ |f|)^p \), \( p > 1 \) (but not if \( p = 1 \)), instead of the measures \( \Delta |f|^p \), \( p > 0 \). In fact, one can easily give a unified result which contains both cases:

THEOREM 2 (WITH J. HYVÖNEN). Let \( H_{2n-1}(E) = 0 \). Let \( f \) be a holomorphic function in \( G \setminus E \). Let \( \varphi: [\infty, \infty) \to \mathbb{R} \) be a nondecreasing function such that \( \varphi|\mathbb{R} \) is strongly convex. Suppose that \( \varphi(\rho, \infty) \) is twice continuously differentiable for some \( \rho \in \mathbb{R} \). If the measure \( \Delta (\varphi \circ \log |f|) \) has locally finite mass near \( E \), then \( f \) has a holomorphic extension to \( G \).

OUTLINE OF PROOF. In view of the proof of Theorem 1 one may restrict to the case \( n = 1 \) (instead of the radius of meromorphy one can as well consider the radius of holomorphy). Take an open set \( D \subseteq G \) and set \( U = \{ z \in D \setminus E | \log |f(z)| > \rho \} \), where \( \rho \in \mathbb{R} \) is as in the theorem. Using a well-known inequality, computing the Laplacian and using the fact that \( \Delta (\varphi \circ \log |f|)(U) \leq \Delta (\varphi \circ \log |f|)(D \setminus E) < \infty \), one gets

\[
\int_{f(U)} \varphi''(\log |w|) \frac{1}{|w|^2} dm_2(w) \leq \int_{U} \varphi''(\log |f(z)|) \frac{|f'(z)|^2}{|f(z)|^2} dm_2(z)
\]

\[
= \Delta (\varphi \circ \log |f|)(U) < \infty.
\]
On the other hand, since \( \varphi \) is strongly convex,
\[
\int_{C \setminus B^2(0, r_0)} \frac{\varphi''(\log |w|)}{|w|^2} \, dm_2(w) = 2\pi \int_{r_0}^{\infty} \frac{\varphi''(\log r)}{r} \, dr = \infty
\]
for each \( r_0 > e^2 \). Thus \( f \) omits a set of positive measure and has by the cited result of Kametani a meromorphic extension to \( D \). Since this extension must be holomorphic, the theorem follows.

REMARK. Proceeding as in [8] one can also consider more general exceptional sets.

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