INARIANT SUBSPACES
FOR ALGEBRAS OF LINEAR OPERATORS
AND AMENABLE LOCALLY COMPACT GROUPS

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ABSTRACT. Let $G$ be a locally compact group. We prove in this paper that $G$ is amenable if and only if the group algebra $L_1(G)$ (respectively the measure algebra $M(G)$) satisfies a finite-dimensional invariant subspace property $T(n)$ for $n$-dimensional subspaces contained in a subset $X$ of a separated locally convex space $E$ when $L_1(G)$ (respectively $M(G)$) is represented as continuous linear operators on $E$. We also prove that for any locally compact group, the Fourier algebra $A(G)$ and the Fourier Stieltjes algebra $B(G)$ always satisfy $T(n)$ for each $n = 1, 2, \ldots$.

1. Introduction. Let $E$ be a separated locally convex space and $X$ a subset of $E$ containing an $n$-dimensional subspace. In [4], K. Fan obtained the following finite-dimensional invariant subspace property $P(n)$ for $n$-dimensional subspaces contained in $X$: If $S = \{T_s : s \in S\}$ is a representation of a left amenable (discrete) semigroup $S$ as continuous linear operators from $E$ into $E$ such that $T_s(L)$ is an $n$-dimensional subspace contained in $X$ whenever $L$ is one and $s \in S$, and there exists a closed $S$-invariant subspace $H$ in $E$ of codimension $n$ with the property that $(x + H) \cap X$ is compact convex for each $x \in E$, then there exists an $n$-dimensional subspace $L_0$ contained in $X$ such that $T_s(L_0) = L_0$ for all $s \in S$. Conversely, as proved by Lau [10] (see also [12]), if $S$ satisfies $P(1)$, then $S$ is left amenable.

In this paper, we prove among other things that (Theorem 3.5) if $G$ is an amenable locally compact group, then the group algebra $L_1(G)$ and measure algebra $M(G)$ satisfy a similar finite-dimensional invariant subspace property $T(n)$ for $n$-dimensional subspaces contained in a subset $X$ of a separated locally convex subspace $E$ when $L_1(G)$ (or $M(G)$) is represented as continuous linear operators on $E$. Conversely, if $L_1(G)$ or $M(G)$ satisfies $T(1)$, then $G$ is amenable. We also show that (Theorem 3.6) if $G$ in any locally compact group, then both the Fourier algebra $A(G)$ and Fourier Stieltjes algebra $B(G)$ (see Eymard [3]) always satisfy $T(n)$ for each $n = 1, 2, \ldots$.

For convenience and to avoid repetition of arguments, we find it most natural to phrase our main result (Theorem 3.4) in terms of $F$-algebras (see [11]). Definition and information on $F$-algebras needed for the proof of our main result will be gathered in §2.
2. \textit{F-algebras.} By an \textit{F-algebra} we shall mean a complex Banach algebra $A$ such that $A^*$ is a $C^*$-algebra and the identity of $A^*$, denoted by $I$ (which always exists \cite[Proposition 1.6.1]{15}), is a multiplicative linear functional on $A$. Examples of $F$-algebras include the group algebra $L^1(G)$, Fourier algebra $A(G)$ and the Fourier Stieltjes algebra $B(G)$ of a locally compact group $G$ (see \cite{11} for details). It also includes the measure algebra $M(S)$ of a locally compact semigroup $S$.

Let $S_A$ denote the set of all positive functionals in $A \subset A^{**}$ with norm one. Then $S_A = \{\mu \in A; \|\mu\| = I(\mu) = 1\}$ \cite[p. 9]{15}. Hence, as readily checked, $(S_A, \ast)$, where $\ast$ denotes the multiplication of $A$, is a semigroup. $A$ is called \textit{left amenable} if $A^* = M$ has a topological left invariant mean (abbreviated as TLIM), i.e. an $m \in M^*$ such that $\|m\| = 1$, $m \geq 0$ and $m(F \cdot \mu) = m(F)$ for each $\mu \in S_A$ and $F \in M$, where $F \cdot \mu \in M$ is defined by $\langle F \cdot \mu, \nu \rangle = \langle F, \mu \ast \nu \rangle$ for all $\nu \in A$.

A function $f \in CB(S_A)$ (the continuous bounded functions on $S_A$), is called \textit{additively uniformly continuous on $S_A$} if given $\varepsilon > 0$, there is some $\delta > 0$ such that $\mu, \nu \in S_A$ and $\|\mu - \nu\| < \delta$ implies $|f(\mu) - f(\nu)| < \varepsilon$. Let $AUC(S_A)$ denote the space of all such functions. It is straightforward to show that $AUC(S_A)$ is a norm closed translation invariant subspace of $CB(S_A)$ containing constants and restrictions of elements in $A^*$ to $S_A$. A linear functional $m \in AUC(S_A)^*$ is called a left invariant mean if $\|m\| = m(1) = 1$ and $m(l_\mu f) = m(f)$, $\forall f \in AUC(S_A)$, $\mu \in S_A$, where $(l_\mu f)(\nu) = f(\mu \ast \nu)$ for all $\nu \in S_A$. The following lemma, which we shall need, is due to Ganeson \cite{5} for the case $A = L^1(G)$ of a locally compact group. Because part of his proof depends heavily on the locally compact group structure, we include a simple proof here for completeness.

**Lemma 2.1.** The following statements are equivalent for an $F$-algebra $A$:

(a) $A$ is left amenable.
(b) $AUC(S_A)$ has a left invariant mean.

**Proof.** (a) implies (b). By \cite[Theorem 4.6]{11}, there is a net $\mu_\alpha \in S_A$ such that $\mu \ast \mu_\alpha - \mu_\alpha \to 0$ in the norm topology for each $\mu \in S_A$. For each $\alpha$, define $m_\alpha(f) = f(\mu_\alpha)$ for each $f \in AUC(S_A)$. Let $m$ be a weak*-cluster point of $\{m_\alpha\}$. Then $m$ is a left invariant mean on $AUC(S_A)$.

(b) implies (a). Define $\tau: A^* \to CB(S_A)$ by $\tau(F)(\mu) = F(\mu)$, $F \in A^*$, $\mu \in S_A$. Then $\tau$ is a continuous linear map of $A^*$ into $CB(S_A)$ such that $\tau \geq 0$, $\tau(1) = 1$ and $\tau(F \cdot \mu) = l_\mu(\tau(F))$ where $\mu \in S_A$, $F \in A^*$ and $l_\mu$ is the left translation operator in $CB(S_A)$. Moreover, $\tau(F) \in AUC(S_A)$. Hence $\tau^*: AUC(S_A)^* \to A^{**}$. If $m$ is a left invariant mean on $AUC(S_A)$, then $\tau^*(m)$ is a TLIM on $A^*$.

3. \textit{Algebra of operators.} A representation of an $F$-algebra $A$ as operators in a locally convex space $E$ is a map $T: A \times E \to E$ denoted by $(\mu, x) \to T_\mu(x)$ such that (1) $T_\mu: E \to E$ is continuous and linear, (2) $\mu \to T_\mu(x)$ is continuous and linear with respect to the norm topology in $A$ for each $x \in E$ and (3) $T_{\mu \ast \nu} = T_\mu \circ T_\nu$, $\forall \mu, \nu \in A$, where $\ast$ denotes multiplication in $A$. Also, let $X$ be a subset of $E$ containing an $n$-dimensional subspace. As in Lau \cite{10}, $\mathcal{L}_n(X)$ denotes all $n$-dimensional subspaces of $E$ contained in $X$. We say that $\mathcal{L}_n(X)$ is $S_A$-invariant under $T$ if $T_\mu(L) \in \mathcal{L}_n(X)$ for each $L \in \mathcal{L}_n(X)$ and $\mu \in S_A$. A closed subspace $H$ in $E$ is called $S_A$-invariant under $T$ if $T_\mu(H) \subset H$, $\forall \mu \in S_A$ (and hence $\forall \mu \in A$ as well). Denote by $q: E \to E/H$ the natural map such that $q(x) = \bar{x}$, $x \in E$. 
**Lemma 3.1.** Let $T: A \times E \to E$ be a representation of an $F$-algebra $A$ and $H$ a closed $S_A$-invariant subspace of $E$ of codimension $n$. Also let $X$ be a subset of $E$ such that $(x + H) \cap X$ is compact and convex for each $x \in E$. Denote by $K$ the set of all linear maps $B \in \mathcal{L}(F,E)$ such that $B(y) \in q^{-1}(y) \cap X \forall y \in F$. If $\mathcal{L}_n(X)$ is nonempty and $S_A$-invariant, then for each $\mu \in A$, the map $T_\mu: E \to E$ induces a map $\tilde{T}_\mu: F \to F$ where $F = E/H$ such that $q \circ T_\mu = \tilde{T}_\mu \circ q \forall \mu \in A$. Moreover, if $\mu \in S_A$, $\tilde{T}_\mu$ is an isomorphism of $F$ onto itself and $K$ is convex and compact in the separated locally convex space $\mathcal{L}(F,E)$ with the topology $\tau$ of pointwise convergence. Defining $\psi: S_A \times K \to K$ by $\psi_\mu(B) = T_\mu \circ B \circ \tilde{T}_\mu^{-1}$, $\mu \in S_A$, $B \in K$, then $\psi$ is an affine action of the semigroup $S_A$ on $K$.

**Proof.** The proof is basically contained in Fan [4, Theorem, p. 447]. We need only consider the semigroup $S_A$ of linear operators in $E$. Note that $\tilde{T}_\mu$ is defined for $\mu \in A$ (not just in $S_A$). However $\tilde{T}_\mu$ need not be an isomorphism unless $\mu \in S_A$.

**Definition 3.2.** The action of $S_A$ on $E$ is called inversely equicontinuous modulo $H$ if given any neighborhood $U$ in $E$, there is some neighborhood $V$ in $E$ such that $V \subset T_\mu(U) + H$ for any $\mu \in S_A$. This is equivalent to the condition that the family $\{\tilde{T}_\mu^{-1}: \mu \in S_A\}$ is equicontinuous on $F$.

**Lemma 3.3.** Under the hypothesis of Lemma 3.1 and if $A$ is left amenable and the action of $S_A$ on $E$ is inversely equicontinuous modulo $H$, then the induced action $\hat{T}: S_A \times F \to F$, where $(\mu, y) \to \hat{T}_\mu y$, $\mu \in S_A$ and $y \in F$, is similar to a unitary representation of the semigroup $S_A$ on the Hilbert space $F$. Moreover, the representation functions $\mu \to (\hat{T}_\mu y, z), y, z \in F$, are in $\text{AUC}(S_A)$. The same is true for the inverse (anti)representation $\{\hat{T}_\mu^{-1}: \mu \in S_A\}$.

**Proof.** Fix a basis $\{\hat{e}_1, \ldots, \hat{e}_n\}$ in $F$ and define $(y, z) = \sum_{j=1}^{\infty} \alpha_j \hat{e}_j$ where $y = \sum_{j=1}^{\infty} \alpha_j \hat{e}_j$ and $z = \sum_{j=1}^{\infty} \beta_j \hat{e}_j$ are in $F$, and let $\|\cdot\|$ denote the induced (Euclidean) norm. Consider the adjoint (anti)representation of $S_A$ on $F$, i.e. $\hat{T}^*: S_A \times F \to F$ such that $(\mu, y) \to \hat{T}_\mu^* y$ where $\mu \in S_A$, $y \in F$ and $\hat{T}_\mu^*$ is the adjoint of $\hat{T}_\mu$ on the Hilbert space $F$. Since the map $\mu \to \hat{T}_\mu y$ is continuous and conjugate linear on $A$ and since

$$\|\hat{T}_\mu^* y\| - \|T_\nu^* y\| \leq \|\hat{T}_\mu^* y - \hat{T}_\nu^* y\| \leq \|\mu - \nu\| \cdot M(y) \quad \forall \mu, \nu \in S_A,$$

where $M(y) \geq 0$, it follows that the function $\mu \to \|\hat{T}_\mu^* y\|$ and hence $\mu \to \|\hat{T}_\mu^* y\|^2$ on $S_A$ is in $\text{AUC}(S_A)$. By the Polarisation Principle, the function $\mu \to (\hat{T}_\mu^* y, \hat{T}_\nu^* z)$ (restricted to $S_A$) is also in $\text{AUC}(S_A)$. Let $m$ be a left invariant mean on $\text{AUC}(S_A)$ (by Lemma 2.1). Define a new inner product on $F$ by $(y, z) = \langle m(\mu), (\hat{T}_\mu^* y, \hat{T}_\mu^* z) \rangle$. Now the family $\{\hat{T}_\mu^*: \mu \in S_A\}$ is pointwise bounded on $F$, hence bounded on $F$ (by the Principle of Uniform Boundedness). On the other hand, inverse equicontinuity of $S_A$ modulo $H$ implies the family $\{\hat{T}_\mu^{-1}: \mu \in S_A\} = \{\hat{T}_\mu^{-1}\}^*: \mu \in S_A\}$ is equicontinuous hence uniformly bounded on $F$. So the family $\{((\hat{T}_\mu^*)^{-1}: \mu \in S_A\} = \{((\hat{T}_\mu^{-1})^*: \mu \in S_A\}$.

Therefore there exist $M, N > 0$ such that

$$N^2\|y\|^2 \leq \inf \{\|\hat{T}_\mu^*(y)\|^2: \mu \in S_A\} \leq \langle m(\mu), \|\hat{T}_\mu^*(y)\|^2 \rangle = \|y\|^2 \leq \sup \{\|\hat{T}_\mu^*(y)\|^2: \mu \in S_A\} \leq M^2\|y\|^2,$$
where $|y|^2 = [y, y]$. Consequently, the two norms are equivalent and, hence, as is well known (see [7]), there is an invertible bicontinuous selfadjoint linear operator $Q \in \mathcal{B}(F)$ such that $[y, z] = (Qy, Qz) \forall y, z \in F$. Now $\forall \mu \in S_A$,

$$
||Q\hat{T}_\mu Q^{-1}(y)||^2 = |\hat{T}_\mu Q^{-1}(y)|^2 = (m(\nu), ||\hat{T}_\mu Q^{-1}(y)||^2)
$$

$$
= (m(\nu), ||\hat{T}_\mu Q^{-1}(y)||^2) = (m(\nu), ||Q^{-1}(y)||^2)
$$

by left invariance of $m$. Therefore $U_\mu = Q\hat{T}_\mu Q^{-1}$ is unitary for any $\mu \in S_A$. This implies that $\{\hat{T}_\mu : \mu \in S_A\}$ is similar to a unitary representation on $F$. Finally, since $\mu \rightarrow (\hat{T}_\mu y, z)$ is bounded linear on $A$, its restriction to $S_A$ is in $\text{AUC}(S_A)$. To prove that the same is true for the inverse representation, we first observe that since $F$ is finite dimensional, the strong operator topology and the uniform operator topology agree on $\mathcal{B}(F)$. Hence $\mu \rightarrow \hat{T}_\mu^*$ is also continuous when $B(F)$ has the uniform operator topology. Consequently given $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu, \nu \in A, ||\mu - \nu|| < \delta$ implies $||\hat{T}_\mu - \hat{T}_\nu|| < \varepsilon$. In particular, if $\mu, \nu \in S_A$ and $||\mu - \nu|| < \delta$, then $||U_\mu - U_\nu|| = ||Q\hat{T}_\mu Q^{-1} - Q\hat{T}_\nu Q^{-1}|| \leq ||Q|| \cdot ||Q^{-1}|| \cdot \varepsilon$. But $\hat{T}_\mu^{-1} = Q\hat{T}_\mu Q^{-1}$ $\forall \mu \in S_A$, since $U_\mu$ is unitary for such $\mu$ and $Q$ is selfadjoint. Therefore

$$
||\hat{T}_\mu^{-1} - \hat{T}_\nu^{-1}|| \leq ||Q||^2 \cdot ||Q^{-1}||^2 \cdot \varepsilon \quad \forall \mu, \nu \in S_A
$$

with $||\mu - \nu|| < \delta$. This implies that the function $\mu \rightarrow y^*\hat{T}_\mu^{-1}y$ on $S_A$ is in $\text{AUC}(S_A)$ $\forall y \in F, y^* \in F^*$.

**Theorem 3.4.** Let $A$ be an $F$-algebra.

(a) If $A$ is left amenable, then $A$ satisfies property $T(n)$ for $n = 1, 2, \ldots$ where property $T(n)$ is defined as follows: Let $E$ be a separated locally convex space and $T : A \times E \rightarrow E$ be a representation of $A$ as linear operators in $E$. Let $X$ be a subset of $E$ such that there exists a closed $S_A$-invariant subspace $H$ of $E$ of codimension $n$ and $(x + H) \cap X$ is compact convex for each $x \in E$. If the action of $S_A$ on $E$ is inversely equicontinuous modulo $H$ and $\mathcal{L}_n(X)$ is nonempty and $S_A$-invariant, then there exists $L_0 \in \mathcal{L}_n(X)$ such that $T_\mu(L_0) = L_0$ $\forall \mu \in S_A$.

(b) If $S$ satisfies property $T(1)$, then $A$ is left amenable (hence $A$ satisfies $T(n)$ for every $n$).

**Proof.** (a) With the notation of Lemma 3.1, define the affine representation $\Psi : S_1 \times K \rightarrow K$ by $\Psi_\mu(B) = T_\mu \circ B \circ \hat{T}_\mu^{-1}$, $\mu \in S_A$ and $B \in K$. We want to show that $\Psi$ is an $A$-representation for the pair $S_A$ and $\text{AUC}(S_A)$ in the sense of Argabright [1, §2]. That is, for each $h \in A(K)$, the affine continuous functions on $K$, the function $\mu \rightarrow h(\Psi_\mu(B))$ is in $\text{AUC}(S_A)$ for each $B \in K$. By Argabright [1, Lemma 1] and Kelley and Namioka [9, Theorem 14.6, p. 120], it is enough to consider $h \in A(K)$ of the form $h(B) = x^*B y$, $B \in K$, where $x^* \in E^*$ and $y \in F$. Let $\{e_1, \ldots, e_n\}$ be a basis in $F$. Then $\{qT_\mu B e_1, \ldots, qT_\mu B e_n\}$ is also a basis in $F$ for any $\mu \in S_A$ and $B \in K$ (by definition of $K$, $S_A$-invariance of $\mathcal{L}_n(X)$ and the fact that $(x + H) \cap X$ is compact and convex for each $x \in E$). Write $y = \sum_{j=1}^{n} \alpha_j(\mu)qT_\mu B e_j$ where $\mu \in S_A, B \in K$. Note that the scalars $\alpha_j(\mu)$ depend only on $\mu \in S_A$ and not on $B$. (See Fan [4, equation (7), p. 449].) Then $\hat{T}_\mu^{-1}(y) = \sum_{j=1}^{n} \alpha_j(\mu)e_j, \mu \in S_A$. (Since $q \circ B = \text{identity on } F$.) By Lemma 3.3, the functions $\mu \rightarrow \alpha_k(\mu) = y_k^*\hat{T}_\mu^{-1}(y)$ on
Sa belong to AUC(Sa) where \(\{y_1^*, \ldots, y_n^*\}\) is the basis dual to \(\{\tilde{e}_1, \ldots, \tilde{e}_n\}\). Now

\[
h(\Psi_\mu(B)) = x^*(T_\mu \circ B \circ \tilde{T}_\mu^{-1})y = \sum_{j=1}^{n} \alpha_j(\mu)x^*T_\mu B(\tilde{e}_j), \quad \mu \in S_A.
\]

Therefore the function \(\mu \mapsto h(\Psi_\mu(B))\) on \(S_A\) is in AUC(Sa). Hence \(\Psi: S_1 \times K \to K\) is an \(A\)-representation for the pair \(S_1\), AUC(Sa). By Argabright [1, Theorem 1], \(\Psi\) has a common fixed point \(Q_0 \in K\). Put \(L_0 = Q_0(F) \in L_n(X)\), then \(T_\mu(L_0) = L_0\ \forall \mu \in S_A\).

(b) Define \(E = A^{**}\) with the weak* topology and \(T: A \times E \to E\) by \(T_\mu(m) = l_\mu^*m\) where \(l_\mu^*\) is the adjoint of the map \(l_\mu: A^* \to A^*\) such that \(l_\mu(F)(\nu) = F(\mu * \nu)\), \(\nu \in A\). As in Lau [10], let \(X\) be the union of all one-dimensional subspaces of \(E = A^{**}\) generated by the means (states) on \(A^*\) and \(H = \{m \in A^{**}: m(I) = 0\}\). The arguments used in the proof of Lau [7, Theorem 1(b)] show that the hypotheses in property \(T(1)\) are all satisfied except the part on inverse equicontinuity modulo \(H\) of \(S_a\), which is also satisfied because the induced action \(\tilde{T}_\mu\) is independent of \(\mu \in S_A\). \((l_\mu(I) = I\ if \ \mu \in S_A, \ since \ I(\mu) = 1, \ if \ \mu \in S_A)\). Since \(S\) satisfies \(T(1)\), this means \(l_\mu^*(L_0) = L_0\ \forall \mu \in S_A\), where \(L_0\) is a one-dimensional subspace generated by some mean \(m_0\). Necessarily, \(m_0\) is a topological left invariant mean on \(A^*\).

A locally compact group \(G\) is amenable if the space \(\text{CB}(G)\) has a left invariant mean. Examples of amenable locally compact groups include all solvable groups, abelian groups and all compact groups. However, if \(G\) contains the free group on two generators as a closed subgroup, then \(G\) is not amenable (see Greenleaf [7] or Pier [13] for details).

**Theorem 3.5.** Let \(G\) be a locally compact group. If \(G\) is amenable, then the group algebra \(L_1(G)\) and the measure algebra \(M(G)\) satisfy \(T(n)\) for each \(n = 1, 2, 3, \ldots\). Conversely, if either \(L_1(G)\) or \(M(G)\) satisfies \(T(1)\), then \(G\) is amenable.

**Proof.** It follows from Greenleaf [7, Theorem 2.2.1] and Wong [17, Theorem 3.3] that amenability of \(G\) is equivalent to the left amenability of \(L_1(G)\) and \(M(G)\).

**Theorem 3.6.** Let \(G\) be a locally compact group. Then both the Fourier algebra \(A(G)\) and the Fourier Stieltjes algebra \(B(G)\) have property \(T(n)\) for each \(n = 1, 2, 3, \ldots\).

**Proof.** In this case, if \(A = A(G)\) or \(B(G)\), the \(A\) is a commutative \(F\)-algebra (see Eymard [3]). Hence \(S_A\) is commutative and so \(l_\infty(S_A)\), the space of bounded complex-valued functions on the discrete semigroup \(S_A\), has an invariant mean (see Day [2]). So by Lemma 2.1, \(A\) is left amenable. Now apply Theorem 3.4.

Given a semigroup \(S\), let \(l_1(S)\) be the Banach algebra as defined in Day [2] or Hewitt and Zuckermann [8]. \(S\) is called left amenable if \(l_\infty(S)\) has a left invariant mean (see Day [2]).

**Theorem 3.7.** Let \(S\) be a semigroup. If \(S\) is left amenable, then the Banach algebra \(l_1(S)\) satisfies \(T(n)\) for each \(n = 1, 2, 3, \ldots\). Conversely if \(l_1(S)\) satisfies \(T(1)\), then \(S\) is left amenable.

**Proof.** Any left invariant mean on \(l_\infty(S)\) is necessarily topological left invariant. Now apply Theorem 3.4 again.
REMARKS. (a) Left amenability of the Fourier algebra $A(G)$ of a locally compact group $G$ was proved by Renauld [14, Theorem 4] (see also Granirer [6, Proposition 5(a)]) by a completely different method.

(b) Theorem 3.7 can be generalized to all locally compact semigroups $S$ with left amenability of $S$ replaced by topological left amenability of the measure algebra $M(S)$ (see [16]), and $l_1(S)$ replaced by $M(S)$.

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