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THE HILL-PENROSE-SPARLING CR MANIFOLDS

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ABSTRACT. This paper gives a simple formulation of the CR structures of Hill, Penrose, and Sparling. An elementary power series argument shows that they cannot be realized as hypersurfaces in an ambient complex manifold.

Suppose M is a smooth three-dimensional real manifold equipped with a nowhere vanishing complex vector field X and a smooth complex-valued function g such that, even locally, the equation $Xf = g$ has no solutions. One may regard X as defining a CR structure on M and then g represents a nowhere vanishing class in the $\bar{\partial}_b$-cohomology $H^{0,1}(M)$. As described in [**2**], this class may be exponentiated to a CR line bundle over M with total space T. This example T of a CR manifold is due to Hill, Penrose, and Sparling who have shown (see [**2**]) that it cannot be locally realized as a hypersurface in $\mathbf{C}^3$. (An alternative proof of this fact is given by Jacobowitz [**1**].)

The manifold T may be defined as $M \times \mathbf{C}$ with CR structure induced by the vector fields

$$X + gz\partial/\partial z \quad \text{and} \quad \partial/\partial\bar{z},$$

where z is the usual coordinate on $\mathbf{C}$. To show that this is nonrealizable, first notice that if a smooth function α satisfies $X\alpha = ng\alpha$, where n is a nonzero constant, then α must be identically zero for, otherwise, any local choice of $n^{-1}\log\alpha$ would provide a solution of $Xf = g$. Suppose that h is CR holomorphic on T. In other words,

$$Xh + gz\partial h/\partial z = 0 = \partial h/\partial\bar{z}.$$

The second equation says that this function is holomorphic in the variable z and so may be expanded as a Taylor series $h = \sum h_j(x)z^j$ with coefficients h_j depending smoothly on $x \in M$. One may apply X term by term and the second equation yields

$$Xh_j + jgh_j = 0 \quad \text{for all } j.$$

As noticed above, the forces $h_j \equiv 0$ for $j \geq 1$. Thus, h is independent of z: $h = h(x)$ for $h(x)$ a CR function on M. Hence, the CR functions fail to separate points as they would do if T were a hypersurface in $\mathbf{C}^3$.

As noted by Jacobowitz [**1**] the CR manifold M need not be restricted to have dimension three. The only requirement is that $H^{0,1}(M)$ be nonzero and the above argument easily generalizes to cover this case. For example, M can be taken to be a hypersurface of Levi type $(1, k)$. The manifold $T = M \times \mathbf{C}$ is always Levi flat in the $\mathbf{C}$ direction.

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