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CONVERGENCE AND INTEGRABILITY OF DOUBLE TRIGONOMETRIC SERIES WITH COEFFICIENTS OF BOUNDED VARIATION

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ABSTRACT. We prove that if $c(j,k) \to 0$ as $\max(|j|,|k|) \to \infty$ and

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11} c(j,k)| < \infty,$$

then the series $\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c(j,k) e^{i(jx+ky)}$ converges both pointwise for every $(x,y) \in (T \backslash \{0\})^2$ and in the $L^p(T^2)$-metric for $0 < p < 1$, where T is the one-dimensional torus. Both convergence statements remain valid for the three conjugate series under these same coefficient conditions.

1. Introduction. We will consider double trigonometric series of the form

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c(j,k) e^{i(jx+ky)} \tag{1.1}$$

where $\{c(j,k)\colon -\infty < j,k < \infty\}$ is a null sequence of complex numbers. Here and in the sequel,

$$(x,y) \in T^2 := \{(x,y) \in R^2 \colon 0 \leq x,\ y < 2\pi\}$$

the two-dimensional torus, whereas $T := \{x \in R\colon 0 \leq x < 2\pi\}$.

The pointwise *convergence* of series (1.1) is usually defined *in Pringsheim's sense* (see, e.g. [**5**, Vol. 2, Chapter 17]). This means that we form the symmetric partial sums

$$s_{MN}(x,y) := \sum_{j=-M}^{M} \sum_{k=-N}^{N} c(j,k) e^{i(jx+ky)} \qquad (M, N \geq 0)$$

and then let both M and N tend to ∞, independently of one another, and assign the limit $f(x,y)$ (if it exists) to series (1.1) as its sum.

Following Hardy [**1**], series (1.1) is said to be *regularly convergent* if

(i) it converges in Pringsheim's sense, and

(ii) the single series

$$\sum_{j=-\infty}^{\infty} c(j,k) e^{i(jx+ky)}$$

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converges for each fixed value of k, and the single series

$$\sum_{k=-\infty}^{\infty} c(j,k)e^{i(jx+ky)}$$

converges for each fixed value of j.

As is known, if (i) and (ii) are satisfied, then the sum $f(x,y)$ of series (1.1) can be computed in the way of iterated summations, too:

$$\begin{aligned} f(x,y) &= \sum_{j=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} c(j,k)e^{i(jx+ky)}\right] \\ &= \sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{\infty} c(j,k)e^{i(jx+ky)}\right]. \end{aligned}$$

The notion of regular convergence was rediscovered in [**2**] where it was defined by the following equivalent condition (and called "convergence in a restricted sense"): the sums

$$\sum_{M_1\le|j|\le M_2}\ \sum_{N_1\le|k|\le N_2} c(j,k)e^{i(jx+ky)} \to 0$$

as $\max\,(M_1,N_1) \to \infty$, independently of the choices of M_2 and N_2 where $0 \le M_1 \le M_2$ and $0 \le N_1 \le N_2$.

Now we introduce an even stronger notion of convergence, we may call it *strongly regular convergence*, which requires that the sums (with $M_1 \le M_2$ and $N_1 \le N_2$)

$$s(Q;x,y) := \sum_{j=M_1}^{M_2}\sum_{k=N_1}^{N_2} c(j,k)e^{i(jx+ky)} \to 0 \tag{1.2}$$

in each of the following limiting cases:

$$\begin{cases} \text{(i)} & M_1 \to \infty, \text{ while } M_2, N_1, \text{ and } N_2 \text{ are arbitrary.} \\ \text{(ii)} & M_2 \to -\infty, \text{ while } M_1, N_1, \text{ and } N_2 \text{ are arbitrary.} \\ \text{(iii)} & N_1 \to \infty, \text{ while } N_2, M_1, \text{ and } M_2 \text{ are arbitrary.} \\ \text{(iv)} & N_2 \to -\infty, \text{ while } N_1, M_1, \text{ and } M_2 \text{ are arbitrary.} \end{cases} \tag{1.3}$$

Here and in the sequel, Q denotes the set of the lattice points of the plane contained in the rectangle $\{(j,k)\colon M_1 \le j \le M_2 \text{ and } N_1 \le k \le N_2\}$.

After these convergence notions, we repeat the definitions of the three conjugate series to (1.1):

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} (-i\,\mathrm{sgn}\, j)c(j,k)e^{i(jx+ky)} \tag{1.4}$$

(*conjugate with respect to x*),

$$\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} (i\,\mathrm{sgn}\, k)c(j,k)e^{i(jx+ky)} \tag{1.5}$$

(*conjugate with respect to y*),

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-i \operatorname{sgn} j)(-i \operatorname{sgn} k) c(j,k) e^{i(jx+ky)} \tag{1.6}$$

(*conjugate with respect to x and y*).

If series (1.4)–(1.6) converge in Pringsheim's sense, then their sums are denoted by $\tilde{f}^{(1,0)}(x,y)$, $\tilde{f}^{(0,1)}(x,y)$, and $\tilde{f}^{(1,1)}(x,y)$, respectively, and are called the corresponding *conjugate functions* to $f(x,y)$ (see, e.g., [**3**]).

2. Main results. Let $\{c(j,k)\colon -\infty < j,k < \infty\}$ be a double sequence. We remind the reader that its differences are defined by

$$\begin{aligned}
\Delta_{10} c(j,k) &= c(j,k) - c(j+1,k),\\
\Delta_{01} c(j,k) &= c(j,k) - c(j,k+1),\\
\Delta_{11} c(j,k) &= c(j,k) - c(j+1,k) - c(j,k+1) + c(j+1,k+1),
\end{aligned}$$

and $\{c(j,k)\}$ is said to be *of bounded variation* if

$$C_{11} := \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11} c(j,k)| < \infty. \tag{2.1}$$

We will prove the following convergence statements.

THEOREM. *Let $\{c(j,k)\colon -\infty < j,k < \infty\}$ be a double sequence of complex numbers that is of bounded variation and such that*

$$c(j,k) \to 0 \quad \text{as } \max(|j|,|k|) \to \infty. \tag{2.2}$$

Then series (1.1)

(i) *converges pointwise in the strongly regular sense to some function $f(x,y)$ for every $(x,y) \in (T\backslash\{0\})^2$;*

(ii) *converges in the $L^p(T^2)$-metric to f for every $0 < p < 1$:*

$$\left\| \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} c(j,k) e^{i(jx+ky)} - f(x,y) \right\|_p \to 0 \tag{2.3}$$

as $M_2, N_2 \to \infty$ and $M_1, N_1 \to -\infty$. In particular, we have $f \in L^p(T^2)$ for every $0 < p < 1$.

(iii) *Analogous conclusions can be drawn for the conjugate series* (1.4)–(1.6) *and the corresponding conjugate functions $\tilde{f}^{(1,0)}$, $\tilde{f}^{(0,1)}$, $\tilde{f}^{(1,1)}$, respectively.*

Here and in the sequel, $\|\cdot\|_p$ denotes the $L^p(T^2)$-norm defined by

$$\|g\|_p := \left[\int_0^{2\pi} \int_0^{2\pi} |g(x,y)|^p \, dx \, dy \right]^{1/p}.$$

Our Theorem can be considered the extension of a theorem of Uljanov [**4**] from single to double trigonometric series.

3. Auxiliary results. We present two lemmas.

LEMMA 1. *If* $\{c(j,k)\}$ *satisfies conditions* (2.1) *and* (2.2), *then for every* k,

$$\sum_{j=-\infty}^{\infty} |\Delta_{10}c(j,k)| \le C_{11}(<\infty), \tag{3.1}$$

$$\sum_{j=-\infty}^{\infty} |\Delta_{10}c(j,k)| \to 0 \quad \textit{as } |k| \to \infty, \tag{3.2}$$

$$\sup_k \sum_{|j|>M} |\Delta_{10}c(j,k)| \to 0 \quad \textit{as } M \to \infty. \tag{3.3}$$

We note that analogous statements are available for the series $\sum_{k=-\infty}^{\infty} |\Delta_{01}c(j,k)|$ under the same conditions.

PROOF. By (2.2),

$$\Delta_{10}c(j,k_0) = \sum_{k=k_0}^{\infty} \Delta_{11}c(j,k),$$

whence

$$\sum_{j=-\infty}^{\infty} |\Delta_{10}c(j,k_0)| \le \sum_{j=-\infty}^{\infty} \sum_{k=k_0}^{\infty} |\Delta_{11}c(j,k)| \tag{3.4}$$

and (3.1) follows.

Again by (2.2),

$$\Delta_{10}c(j,k_0) = -\sum_{k=-\infty}^{k_0-1} \Delta_{11}c(j,k),$$

whence

$$\sum_{j=-\infty}^{\infty} |\Delta_{10}c(j,k_0)| \le \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{k_0-1} |\Delta_{11}c(j,k)|. \tag{3.5}$$

Clearly (3.2) follows from (2.1), (3.4), and (3.5).

Finally, (3.3) is a consequence of (3.2) (applied for large values of $|k|$) and (3.1) (applied for small values of $|k|$).

LEMMA 2. *If* $\{c(j,k)\}$ *satisfies conditions* (2.1) *and* (2.2), *then the sequences*

$$\begin{cases} \text{(i)} & \{(-i\operatorname{sgn} j)c(j,k)\}, \\ \text{(ii)} & \{(-i\operatorname{sgn} k)c(j,k)\}, \\ \text{(iii)} & \{(-i\operatorname{sgn} j)(-i\operatorname{sgn} k)c(j,k)\} \end{cases} \tag{3.6}$$

also satisfy these same conditions.

PROOF. It suffices to prove (i), since (ii) is a symmetric counterpart of (i), while (iii) follows from a repeated application of (i) and (ii). In the case of (i), the fulfillment of (2.2) is obvious. Inequality (2.1) follows from Lemma 1:

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|\Delta_{11}[(-i\operatorname{sgn} j)c(j,k)]| \\
&\quad=\left\{\sum_{j=-\infty}^{-2}\sum_{k=-\infty}^{\infty}+\sum_{j=1}^{\infty}\sum_{k=-\infty}^{\infty}\right\}|\Delta_{11}c(j,k)| \\
&\qquad+\sum_{k=-\infty}^{\infty}|\Delta_{01}c(-1,k)|+\sum_{k=-\infty}^{\infty}|\Delta_{01}c(1,k)|.
\end{aligned}$$

4. Proof of the Theorem. We will use the notation $w(x)=1-e^{-ix}$. Then

$$|w(x)|=2\sin\frac{x}{2}\quad\text{for } 0\le x<2\pi \tag{4.1}$$

and

$$w(x)w(y)=1-e^{-ix}-e^{-iy}+e^{i(x+y)}.$$

Part 1: *Pointwise convergence.* Performing an "Abel transformation-like" rearrangement yields, for any $M_1\le M_2$ and $N_1\le N_2$,

$$\begin{aligned}
&w(x)w(y)\sum_{j=M_1}^{M_2}\sum_{k=N_1}^{N_2}c(j,k)e^{i(jx+ky)} \\
&\quad=\sum_{j=M_1-1}^{M_2-1}\sum_{k=N_1-1}^{N_2-1}\Delta_{11}c(j,k)e^{i(jx+ky)} \\
&\qquad+\sum_{j=M_1-1}^{M_2-1}\Delta_{10}c(j,N_2)e^{i(jx+N_2y)} \\
&\qquad-\sum_{j=M_1-1}^{M_2-1}\Delta_{10}c(j,N_1-1)e^{i(jx+(N_1-1)y)} \\
&\qquad+\sum_{k=N_1-1}^{N_2-1}\Delta_{01}c(M_2,k)e^{i(M_2x+ky)} \\
&\qquad-\sum_{k=N_1-1}^{N_2-1}\Delta_{01}c(M_1-1,k)e^{i((M_1-1)x+ky)} \\
&\qquad+c(M_2,N_2)e^{i(M_2x+N_2y)}.
\end{aligned} \tag{4.2}$$

Hence, using (4.1) and notation (1.2),

$$|s(Q;x,y)| \le \frac{1}{4\sin\frac{x}{2}\sin\frac{y}{2}}\left\{\sum_{j=M_1-1}^{M_2-1}\sum_{k=N_1-1}^{N_2-1}|\Delta_{11}c(j,k)|\right.$$
$$+\sum_{j=M_1-1}^{M_2-1}\Big[|\Delta_{10}c(j,N_2)|+|\Delta_{10}c(j,N_1-1)|\Big]$$
$$\left.+\sum_{k=N_1-1}^{N_2-1}\Big[|\Delta_{01}c(M_2,k)|+|\Delta_{01}c(M_1-1,k)|\Big]+|c(M_2,N_2)|\right\}.$$

Making use of Lemma 1, we can see that each term in the braces on the right-hand side tends to zero as any one of the limiting cases (i)–(iv) in (1.3) is realized. Thus, the sum $f(x,y)$ of series (1.1) certainly exists for all $0 < x, y < 2\pi$.

Part 2: $L^p(T^2)$*-convergence.* It is plain that

$$\text{(4.3)}\qquad |f(x,y)-s(Q;x,y)| \le \left|\sum_{j=-\infty}^{M_1-1}\sum_{k=-\infty}^{\infty}c(j,k)e^{i(jx+ky)}\right|$$
$$+\left|\sum_{j=M_2+1}^{\infty}\sum_{k=-\infty}^{\infty}c(j,k)e^{i(jx+ky)}\right|$$
$$+\left|\sum_{j=M_1}^{M_2}\sum_{k=-\infty}^{N_1-1}c(j,k)e^{i(jx+ky)}\right|$$
$$+\left|\sum_{j=M_1}^{M_2}\sum_{k=N_2+1}^{\infty}c(j,k)e^{i(jx+ky)}\right|$$
$$=: \sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3+\sum\nolimits_4, \quad \text{say.}$$

Similarly to (4.2), for any $M > M_2$ and $N > 0$,

$$\text{(4.4)}\qquad w(x)w(y)\sum_{j=M_2+1}^{M}\sum_{k=-N}^{N}c(j,k)e^{i(jx+ky)}$$
$$=\sum_{j=M_2}^{M-1}\sum_{k=-N-1}^{N-1}\Delta_{11}c(j,k)e^{i(jx+ky)}+\sum_{j=M_2}^{M-1}\Delta_{10}c(j,N)e^{i(jx+Ny)}$$
$$-\sum_{j=M_2}^{M-1}\Delta_{10}c(j,-N-1)e^{i(jx-(N+1)y)}$$
$$+\sum_{k=-N-1}^{N-1}\Delta_{01}c(M,k)e^{i(Mx+ky)}$$
$$-\sum_{k=-N-1}^{N-1}\Delta_{01}c(M_2,k)e^{i(M_2x+ky)}+c(M,N)e^{i(Mx+Ny)}.$$

Letting $M, N \to \infty$, by Lemma 1 we get that

$$w(x)w(y) \sum_{j=M_2+1}^{\infty} \sum_{k=-\infty}^{\infty} c(j,k)e^{i(jx+ky)}$$
$$= \sum_{j=M_2}^{\infty} \sum_{k=-\infty}^{\infty} \Delta_{11}c(j,k)e^{i(jx+ky)} - \sum_{k=-\infty}^{\infty} \Delta_{01}c(M_2,k)e^{i(M_2x+ky)}.$$

Hence

$$\sum\nolimits_2 \le \frac{1}{|w(x)w(y)|} \left\{ \sum_{j=M_2+1}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11}c(j,k)| + \sum_{k=-\infty}^{\infty} |\Delta_{01}c(M_2,k)| \right\}.$$

By (4.1), for $0 < p < 1$,

$$\left\| \frac{1}{w(x)w(y)} \right\|_p = \left[\int_0^{2\pi} \int_0^{2\pi} \frac{dx\,dy}{4^p \sin^p \frac{x}{2} \sin^p \frac{y}{2}} \right]^{1/p} < \infty. \tag{4.5}$$

So, by (2.1) and the counterpart of (3.2) (when the roles of j and k are interchanged), we obtain that

$$\left\| \sum\nolimits_2 \right\|_p \to 0 \quad \text{as } M_2 \to \infty. \tag{4.6}$$

In a similar way, we can deduce that

$$\left\| \sum\nolimits_1 \right\|_p \to 0 \quad \text{as } M_1 \to -\infty. \tag{4.7}$$

Estimating $\sum_4$, let $N > N_2$. Then, following the pattern occurring in (4.2),

$$\begin{aligned}
(4.8)\quad w(x)w(y) &\sum_{j=M_1}^{M_2} \sum_{k=N_2+1}^{N} c(j,k)e^{i(jx+ky)} \\
&= \sum_{j=M_1-1}^{M_2-1} \sum_{k=N_2}^{N-1} \Delta_{11}c(j,k)e^{i(jx+ky)} \\
&+ \sum_{j=M_1-1}^{M_2-1} \Delta_{10}c(j,N)e^{i(jx+Ny)} \\
&- \sum_{j=M_1-1}^{M_2-1} \Delta_{10}c(j,N_2)e^{i(jx+N_2y)} \\
&+ \sum_{k=N_2}^{N-1} \Delta_{01}c(M_2,k)e^{i(M_2x+ky)} \\
&- \sum_{k=N_2}^{N-1} \Delta_{01}c(M_1-1,k)e^{i((M_1-1)x+ky)} + c(M_2,N)e^{i(M_2x+Ny)}.
\end{aligned}$$

Letting $N \to \infty$, by Lemma 1,

$$w(x)w(y) \sum_{j=M_1}^{M_2} \sum_{k=N_2+1}^{\infty} c(j,k)e^{i(jx+ky)} = \sum_{j=M_1-1}^{M_2-1} \sum_{k=N_2}^{\infty} \Delta_{11}c(j,k)e^{i(jx+ky)}$$

$$- \sum_{j=M_1-1}^{M_2-1} \Delta_{10}c(j,N_2)e^{i(jx+N_2y)}$$

$$+ \sum_{k=N_2}^{\infty} \Delta_{01}c(M_2,k)e^{i(M_2x+ky)}$$

$$- \sum_{k=N_2}^{\infty} \Delta_{01}c(M_1-1,k)e^{i((M_1-1)x+ky)}.$$

Hence

$$\sum\nolimits_4 \le \frac{1}{|w(x)w(y)|} \left\{ \sum_{j=M_1-1}^{M_2-1} \sum_{k=N_2}^{\infty} |\Delta_{11}c(j,k)| + \sum_{j=M_1-1}^{M_2-1} |\Delta_{10}c(j,N_2)| + \sum_{k=N_2}^{\infty} |\Delta_{01}c(M_2,k)| + \sum_{k=N_2}^{\infty} |\Delta_{01}c(M_1-1,k)| \right\}.$$

By (2.1), (3.2), and its counterpart, we get that

$$\left\| \sum\nolimits_4 \right\|_p \to 0 \quad \text{as } N_2 \to \infty. \tag{4.9}$$

A similar argument gives that

$$\left\| \sum\nolimits_3 \right\|_p \to 0 \quad \text{as } N_1 \to -\infty. \tag{4.10}$$

Combining (4.3), (4.6), (4.7), (4.9), and (4.10) yields (2.3).

Part 3: *Convergence of the conjugate series.* This immediately follows from (i) and (ii) of the Theorem, via Lemma 2.

References

1. G. H. Hardy, *On the convergence of certain multiple series*, Proc. Cambridge Philos. Soc. **19** (1916–1919), 86–95.
2. F. Móricz, *On the convergence in a restricted sense of multiple series*, Anal. Math. **5** (1979), 135–147.
3. F. Móricz and Xianliang Shi, *Approximation to continuous functions by Cesàro means of double Fourier series and conjugate series*, J. Approx. Theory **49** (1987), 346–377.
4. P. L. Uljanov, *Application of A-integration on a class of trigonometric series*, Mat. Sb. **35(77)** (1954), 469–490. (Russian)
5. A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, New York, 1959.

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