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EXTREMAL LENGTHS ON DENJOY DOMAINS

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ABSTRACT. We consider the problem of computing the extremal lengths of certain homotopy classes of curves in certain symmetric surfaces. Specifically, we concentrate on plane domains which are conformal to the Riemann sphere with a collection of slits in the real axis removed; such a conformal type is called a *Denjoy domain*. Using Jenkins-Strebel forms, the extremal length of any sufficiently symmetric homotopy class of curves is computed in terms of the endpoints of the slits. One can then choose a symmetric pants decomposition of the surface and invert the formulas derived, which are a set of coupled quadratic equations. In this way, one obtains a coordinatization of the space of all marked Denjoy domains of a fixed topological type.

We consider the (marked) conformal type of a domain of the form $F = \mathbb{S}^2 - \Sigma$, where $\mathbb{S}^2$ denotes the Riemann sphere, and $\Sigma \subset \mathbb{S}^1 \subset \mathbb{S}^2$ consists of a finite collection of arcs, called *slits*. The extremal length (defined below) is a number associated with a conformal class of metric and a homotopy class of connected curves in F. In this note, we prove an analogue of the Fricke-Klein Theorem to the effect that extremal lengths of a finite family of simple curve classes uniquely determine conformal type. (This is in contrast to [**HM**], where homotopy classes of families of curves are considered.) This "injectivity question" on a general Riemann surface has been considered in [**Pi**], where the analogue of the Fricke-Klein Theorem is proved for the sphere-minus-four-points.

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Let $F^r = \mathbb{S}^2 - \{\bigcup_{j=1}^{r-1}(2j-3, 2j-2) \cup (2r-3, \infty)\}$ and let Δ_i be a simple closed curve encircling the ith component of $\mathbb{S}^2 - F^r$, $i = 1, \ldots, r$, as in Figure 1. A *pants decomposition* $\mathcal{P}$ of F^r is a collection of simple curves $\{\Gamma_j\}_2^{r-2}$ (as in Figure 1) so that components of $F - \bigcup\{\Gamma_j\}$ have Euler characteristic -1.

A metric μ on F^r will be regarded as a line element $ds = f(z)|dz + g(z)d\bar{z}|$ where $f \geq 0$ with isolated zeros and $\|g\|_\infty < 1$. The *Teichmüller space* $\mathcal{T}^r$ of F^r is the space of conformal classes of finite-area complete metrics on F^r modulo push-forward by diffeomorphisms isotopic to Dehn twists along Δ_i, $i = 1, \ldots, r$. We take $r \geq 3$ so that F^r is equipped with the intrinsic Poincaré metric $\mu_\rho \in \rho \in \mathcal{T}^r$ (see [**Ab**]).

Let $\iota\colon F^r \circlearrowleft$ denote the involutive orientation-reversing homeomorphism induced by conjugation on $\mathbb{S}^2$ and $\iota^*\colon \mathcal{T}^r \circlearrowleft$ the corresponding homeomorphism. A conformal class $\rho \in \mathcal{T}^r$ is said to be a (marked) *Denjoy domain* (and we write

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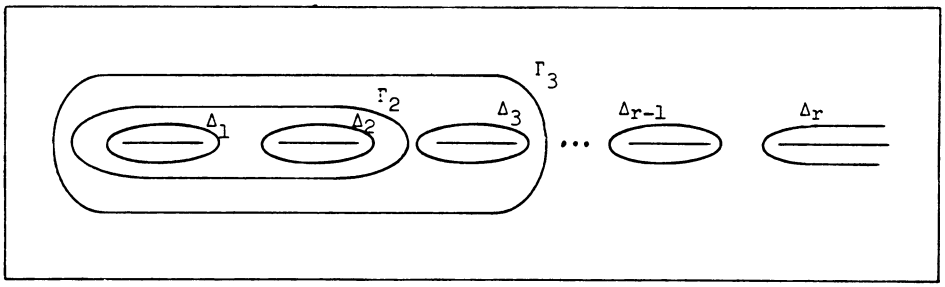

FIGURE 1

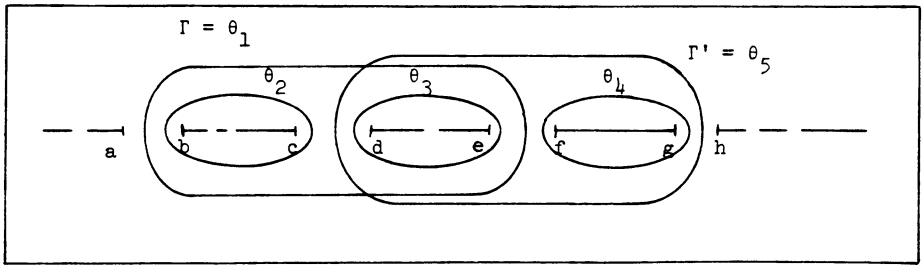

FIGURE 2

$\rho \in \mathcal{D}^r \subset \mathcal{T}^r$) if $\iota^*\rho = \rho$. ($\mathcal{D}^r$ is the real Teichmüller space of type F^r/ι in the sense of [**Ea**].) A pants decomposition $\mathcal{P}$ is *symmetric* if $\iota\mathcal{P}$ is isotopic to $\mathcal{P}$.

Suppose Γ is a homotopy class of simple curve in F^r and μ is a metric on F. We let $\mu(\Gamma) = \inf_{\gamma\in\Gamma}(\mu\text{-length of } \gamma)$ and define the *extremal length* (see [**Ah**]) of Γ for $\rho \in \mathcal{T}^r$ to be

$$\lambda_\rho(\Gamma) = \sup_{\mu\in\rho} \frac{[\mu(\Gamma)]^2}{(\mu\text{-area of } F^r)}.$$

Our main result follows.

THEOREM. *Suppose $\mathcal{P} = \{\Gamma_j\}_2^{r-2}$ is a symmetric pants decomposition. The map*

$$\begin{aligned} \Lambda\colon \mathcal{D}^r &\to \mathbf{R}_+^{2r-3} \\ : \rho &\to (\lambda_\rho(\Delta_i))_1^{r-1} \times (\lambda_\rho(\Gamma_j))_2^{r-2} \end{aligned}$$

is a proper homeomorphism.

It is standard [**HM**] that Λ is continuous on $\mathcal{T}^r$, and insofar as $\mu_\rho(\Gamma) \le \pi\lambda_\rho(\Gamma)$, Mumford's [**Mu**] exhaustion of $\mathcal{T}^r$ shows that Λ is proper. Our work involves showing Λ is bijective.

The link between conformal geometry and topology is provided by quadratic differentials (see [**Je**, **St**]). We recall the

JENKINS-STREBEL THEOREM. *Given an isotopy class Γ of simple curves on F^r and $\rho \in \mathcal{T}^r$, there is a unique projective class of holomorphic quadratic differentials $\omega = \omega_\rho(\Gamma)$ so that all the noncritical trajectories of ω lie in Γ.*

The metric of ω is the extremal metric for $\lambda_\rho(\Gamma)$. The union of the noncritical trajectories of ω forms an annulus $A_\rho(\Gamma)$ whose modulus is the reciprocal of $\lambda_\rho(\Gamma)$.

We will also need the Grötsch function (see [**LV**, p. 53])

$$\begin{aligned}\mu\colon (0,1) &\to \mathbf{R}_+\\ : r &\to \text{modulus of } \mathbf{D}^2 - (0,r),\end{aligned}$$

where $\mathbf{D}^2$ denotes the unit disc in $\mathbf{S}^2$. The key fact [**Ah**] for us is that μ is a homeomorphism. We define the *Teichmüller annulus* $T(a) = \mathbf{S}^2 - ((-1,0) \cup (a,\infty))$, $a \in \mathbf{R}_+$, and recall [**LV**, Theorem II. 1.1] that the modulus of $T(a)$ is $2\mu((1+a)^{-1/2})$. The *Teichmüller width* $x_\rho(\Gamma)$ is the value of $a \in \mathbf{R}_+$ so that $T(a)$ is conformal to $A_\rho(\Gamma)$. It follows that

$$1 + x_\rho(\Gamma) = [\mu^{-1}(1/(2\lambda_\rho(\Gamma)))]^{-2}.$$

The correspondence between Teichmüller widths and extremal lengths is seen to be bijective, and the former will prove more natural for what follows.

Given a sequence $0 < a_1 < \cdots < a_{2r-3}$ of positive reals, we define the Riemann surface $F(a_1,\ldots,a_{2r-3})$ to be $\mathbf{S}^2 - \bigcup_1^r (a_{2j-3}, a_{2j-2})$ equipped with the spherical metric, where we have set $a_{-1} = -1$, $a_0 = 0$, $a_{2r-2} = \infty$ for convenience. It follows from the Riemann mapping theorem that if $\rho \in \mathcal{D}^r$, then there is an orientation-preserving conformal homeomorphism $(F^r, \rho) \to F = F(a_1,\ldots,a_{2r-3})$, which preserves marking in the obvious sense. We identify homotopy classes of curves on F^r with those in F, and let $\iota\colon F \circlearrowleft$ denote the restriction of conjugation on $\mathbf{S}^2$ to F.

LEMMA. *The theorem holds if $\mathcal{P}$ is the standard pants decomposition illustrated in Figure* 1.

PROOF. Suppose that $\rho \in \mathcal{D}^r$ is conformally equivalent to $F = F(a_1,\ldots,a_{2r-3})$, $0 < a_1 < \cdots < a_{2r-3}$. Let

$$\begin{aligned} x_i &= x_\rho(\Delta_i), \qquad i = 1,\ldots,r,\\ y_j &= x_\rho(\Gamma_j), \qquad i = 2,\ldots,r-2.\end{aligned}$$

We claim that the following equalities hold.

$$(*)\qquad \begin{aligned} x_1 &= a_1,\\ x_i &= \frac{(a_{2i-1}-a_{2i-2})(a_{2i-3}-a_{2i-4})}{(a_{2i-1}-a_{2i-4})(a_{2i-2}-a_{2i-3})}, \qquad i = 2,\ldots,r-1,\\ x_r &= \frac{a_{2r-3}-a_{2r-4}}{1+a_{2r-4}},\\ y_j &= \frac{a_{2j-1}-a_{2j-2}}{1+a_{2j-2}}, \qquad j = 2,\ldots,r-2,\end{aligned}$$

where we have again set $a_0 = 0$ for convenience.

We start with the first equation and consider the ring domain $A = A_\rho(\mu)$ in F. Since A is invariant under ι, so too is the critical locus Z of the Jenkins-Strebel form $\omega_\rho(\Gamma)$. Suppose that $\gamma \subset Z$ is a piecewise analytic arc with endpoints in $\mathbf{S}^1 = \partial\mathbf{D}^2$. If $\gamma \not\subset \mathbf{S}^1 - \bigcup_1^r (a_{2j-3}, a_{2j-2})$, then there is a subarc $\gamma' \subset \gamma$ with $\gamma' \cap \mathbf{S}^1 = \partial\gamma' \cap \mathbf{S}^1$ and $\gamma' \cup \iota(\gamma')$ separates $\mathbf{S}^2$. This is a contradiction since $F - Z = A$ is connected. Thus $Z \subset \mathbf{S}^1$, so A is the Teichmüller annulus $T(a_1)$, whence $x_1 = a_1$, as was claimed. Routine computations with Möbius transformations similarly give the other equations $(*)$.

We prove the Lemma by showing that the coupled system of $2r-3$ nonlinear equations $(*)$ defines a bijection between $\{0 < a_1 < \cdots < a_{2r-3} < \infty\}$ and $\mathbf{R}_+^{2r-3}$. This involves computing a_k inductively for $k = 1, \ldots, 2r-3$ from $(x_i) \times (y_j)$, checking at each step existence (independent of $(x_i) \times (y_j)$) and uniqueness of a solution with $a_k > a_{k-1}$, $k = 1, \ldots, 2r-3$, and the case $k = 1$ is trivial.

The cases $k = 2, 3$ on F^3 will be considered separately first. The equation for x_3 gives that $a_3 = x_3 + a_2(1 + x_3)$, and the equation for x_2 gives the quadratic

$$\begin{aligned} 0 &= x_2(1+x_3)a_2^2 + (x_2x_3 - x_1x_2 - x_1x_3 - x_1x_2x_3)a_2 - x_1x_3(1+x_2) \\ &= \alpha a_2^2 + \beta a_2 + \gamma \end{aligned}$$

in a_2. Now, $\alpha > 0$ and $\gamma < 0$, so there is a unique positive solution if and only if the discriminant is positive. One checks that this is automatic for $x_1, x_2, x_3 > 0$ and that $0 < a_1 < a_2 < a_3 < \infty$, and the Lemma follows in this case.

If $r > 3$, suppose that $0 < a_1 < \cdots < a_{2K-3}$ are known and the corresponding $x_1, \ldots, x_{2K-3}$ are unrestricted. We introduce the change of variables

$$a = a_{2K-4}, \quad b = a_{2K-3} - a_{2K-4}, \quad c = a_{2K-2} - a_{2K-4}, \quad d = a_{2K-1} - a_{2K-4},$$

and the equations for x_K and y_K yield the quadratic

$$\begin{aligned} 0 &= x_K(1+y_K)c^2 + [x_Ky_K(1+a) - b(1+y_K) - by_K]c - by_K(1+a)(1+x_K) \\ &= \alpha c^2 + \beta c + \gamma. \end{aligned}$$

Again $\alpha > 0$, $\gamma < 0$, $\beta^2 > 4\alpha\gamma$ hold automatically and $0 < c < d$.

Finally, the computation of a_{2r-4} and a_{2r-3} parallels the computation of a_2 and a_3 on F^3 above, and the Lemma follows. $\square$

We define a move on a symmetric pants decomposition $\mathcal{P}$ as follows. Suppose $\Gamma \in \mathcal{P}$ and Γ' is a ι-invariant curve class disjoint from $\mathcal{P} - \Gamma$ with $\Gamma' \cap \Gamma = 2$ as in Figure 2. We say the symmetric pants decomposition $\mathcal{P}' = \mathcal{P} \cup \Gamma' - \Gamma$ arises from $\mathcal{P}$ by an *exchange*. An easy argument shows that exchanges act transitively on symmetric pants decompositions.

Suppose $\mathcal{P}'$ differs from $\mathcal{P}$ by an exchange and assume the theorem for $\mathcal{P}$. Adopt the notation of Figure 2 and normalize so that $a = \infty$, $b = -1$, and $c = 0$. Computations already done in the Lemma allow one to compute $(x_\rho(\theta_i))_2^5$ uniquely from $(x_\rho(\theta_i))_1^4$, and the theorem follows. $\square$

We have proved somewhat more than the theorem. The transformation on Teichmüller widths is $\mathbf{R}$-algebraic (in the sense that its graph is a $\mathbf{R}$-algebraic variety intersected with the positive orthant). It follows that the centralizer of ι in $\pi_0 \operatorname{Homeo} F^r$ acts $\mathbf{R}$-algebraically on our "Teichmüller width coordinates" on $\mathcal{D}^r$. Moreover, we have explicitly computed Teichmüller widths of symmetric curves in terms of endpoints $a_1, \ldots, a_{2r-3}$.

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